Point Estimation

As an example, consider the Bond-Strength data in Table 2.1, atop page 26 of Montgomery’s 8th edition, on Modified Mortar (the experimental group). The most two important sample statistics for the response variable, \( y_1 \), of the experimental group are

\[
\bar{y}_1 = \frac{1}{n_1} \sum_{j=1}^{10} y_{1j} = 16.764, \quad s_1^2 = \frac{1}{n_1 - 1} \sum_{j=1}^{n_1} (y_{1j} - \bar{y}_1)^2 = 0.10014, \quad \text{where } n_1 = 10
\]

and the unit of measurements is in kgf/cm². Before sampling, \( \bar{y}_1 \) is an unbiased estimator of \( \mu_1 \), i.e., \( E(\bar{y}_1) = \mu_1 \), and \( s_1^2 \) is an unbiased estimator of \( \sigma_1^2 \) iff the population is infinite in which case \( E(s_1^2) = \sigma_1^2 \). If the population is finite, \( E(s_1^2) \neq \sigma_1^2 \). For nearly all underlying distributions, \( E(S) \neq \sigma \), i.e., \( S \) is a biased estimator of population standard deviation \( \sigma \). For a normal universe, \( E(S) = c_4 \sigma \), where \( 0 < c_4 < 1 \), and \( c_4(n) = \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]} \).

The operator \( E \) is linear because

(1): \( E(CY) = CE(Y) \), and (2): \( E(Y_1 + Y_2) = E(Y_1) + E(Y_2) \), where \( C \) is any constant.

The operator \( V \) is nonlinear because

(1): \( V(CY) \neq CV(Y) \). In fact \( V(CY) = C^2V(Y) \) and

\( V(Y_1 \pm Y_2) = V(Y_1) + V(Y_2) \pm 2COV(Y_1, Y_2) \). If \( Y_1 \) and \( Y_2 \) are independent, then

\( COV(Y_1, Y_2) = \sigma_{12} = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = 0 \). The converse of this is not generally true. Now, consider the numerator of \( s_2^2 = \frac{1}{n} \sum_{j=1}^{n} (y_j - \bar{y})^2 / (n - 1) = S_{yy} / (n - 1) = CSS / (n - 1): \)

\[
S_{yy} = \sum_{j=1}^{n} (y_j - \bar{y})^2 = \sum_{j=1}^{n} y_j^2 - 2\bar{y}\sum_{j=1}^{n} y_j + \sum (y)^2 = \sum y_j^2 - (\sum y_j)^2 / n
\]

\[
S_{yy} = \quad CSS = USS - CF
\]

Degrees of freedom (df):

\( (n - 1) = n - 1 \)
In general, if a random variable (rv), \( Y \), has \( V(Y) = \sigma^2_Y \), then \( E(CSS/ df) = \sigma^2_Y \).

**Interval Estimation**

There are 3 types of QCH: Smaller-The-Better (STB), LTB, & Nominal-The-Best (NTB)

**STB Examples:** Tire eccentricity, Loudness of a compressor, Rate of wear, Braking distance, etc → Ideal target = 0 and only a single upper spec-limit, USL = \( y_u \).

**LTB (Larger-The-Better) Examples:** Welding Strength, TTF (time to failure), Efficiency, Yield, etc. → Ideal target = \( \infty \) and a single lower spec-limit, LSL = \( y_L \).

**NTB Examples:** Clearance, Chemical content level, Output voltage, % Asphalt in a hot-mix asphalt (HMA) which generally ranges within 3.5-8.00%. → Ideal target = \( m \) and there are always an LSL = \( m - \Delta_1 \) and an USL = \( m + \Delta_2 \).

**Example 1.** A company manufactures ropes for climbing purposes. The consumers’ LSL for breaking strength \( y \) is \( y_L = 1200 \) psi, and \( y \sim N(\mu, 676 \text{ psi}^2) \). Obtain a 95% proper CI for the parameter \( \mu \) using the average of a random sample of size \( n = 25 \), where \( \bar{y} = 1215 \) psi.

![Figure 1](image-url)
The sampling distribution (SMD) of $\overline{y}$ in Figure 1 shows that the $\text{Pr}(\overline{y} \leq \mu + 1.645 \times \frac{26}{\sqrt{n}}) = 0.95 \rightarrow 1215 - 1.645 \times 5.2 \leq \mu < \infty \rightarrow 1206.446 \leq \mu < \infty$, where $\mu_L = 1206.446$, and $Z_{0.05} = 1.645$.

If a CI is lower one-sided, then the corresponding test of hypothesis must be right-tailed, i.e., for the above CI we should test $H_0: \mu = \mu_0 \text{ psi}$ versus $H_1: \mu > \mu_0$ (the alternative $H_1: \mu < \mu_0$ will lead to a contradiction when $H_0$ is rejected). Typical values of $\mu_0 = 1205, 1208, 1210$, etc.

The nominal level of significance is generally set at $\alpha = 0.05$. $A_U = \text{Upper Acceptance Limit} = 1205 + Z_{0.05} \times 26 / \sqrt{25} = 1213.554 = \overline{y}_U \rightarrow \text{AI (Acceptance Interval)}: 0 < \overline{y} \leq 1213.554$. The test statistic $\overline{y} = 1215 > \overline{y}_U \rightarrow$ Reject $H_0$ at the LOS $\alpha = 0.05$ and conclude that $\mu > 1205$. Note that an upper one-sided CI for the above test is given by $-\infty < \mu \leq 1215 + 8.554 = 1223.554$, which includes the hypothesized value of $\mu = 1205$ and hence contradictory to the rejection of $H_0$.

**Assignment 1.** Work problem 2.17 on page 60 of your text. ANS: (c) $P\text{-value} = 0.0549$, (d) $[799.75, 824.25]$. (b) Work problem 2.20, p. 61. ANS: $n \geq 139$. (c) Work problem 2.5. (d) Work problem 2.22 but change part (a) to determining if the population mean repair time is less than 250 hours? Note that in this problem you will have to use the Student’s $t$-distribution.
**Test of Hypothesis**

In conducting any test of hypothesis, only one of the 4 circumstances given in Table 1 will occur, where Pr denotes probability.

**Table 1. The four circumstances that may occur when testing H₀**

<table>
<thead>
<tr>
<th>H₀ is true</th>
<th>H₀ is false</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reject H₀</td>
<td>Correct decision (True Positive),</td>
</tr>
<tr>
<td></td>
<td>Occurrence Pr = α</td>
</tr>
<tr>
<td>Type I error (or False Positive)</td>
<td>Correct decision (True Positive),</td>
</tr>
<tr>
<td></td>
<td>Occurrence Pr = 1–β = Power or</td>
</tr>
<tr>
<td></td>
<td>Sensitivity of the test</td>
</tr>
<tr>
<td>Accept H₀</td>
<td>Correct decision (True Negative)</td>
</tr>
<tr>
<td></td>
<td>Specificity of the test</td>
</tr>
<tr>
<td></td>
<td>Occurrence Pr = 1–α</td>
</tr>
<tr>
<td>Accept H₀</td>
<td>Type II error (or False Negative)</td>
</tr>
<tr>
<td></td>
<td>Occurrence Pr = β = The Pr of Accepting</td>
</tr>
<tr>
<td></td>
<td>H₀ at a specified value of the parameter</td>
</tr>
<tr>
<td></td>
<td>under H₀.</td>
</tr>
</tbody>
</table>

![Figure 3](image-url)

Note that in order to commit a type II error, H₀ must be false. In reference to Example 1, this implies that μ must differ from 1205 psi, say μ = 1210. From Figure 3, $Z_{1-\beta} = (1213.554 - 1210)/5.2 = 0.683462 \rightarrow \beta(\text{at } \mu = 1210) = \Phi(0.683462) = 0.7528424$. 
**Assignment 2.** Compute the values of β for values of μ = 1205, 1213.554, 1215, 1218 and 1225. Graph β as a function of μ. This graph of type II error Pr versus the parameter under H₀ is called the OC (Operating Characteristic) curve.

If the population variance σ² is unknown, then statistical inference (i.e., estimation & test of hypothesis) on μ cannot be conducted using the statistic \( Z_0 = (\bar{y} - \mu_0) \sqrt{n} / \sigma \), rather resort has to be made to the sampling distribution of the statistic \( (\bar{y} - \mu_0) \sqrt{n} / S \) which has (W. S. Gosset’s) Student’s t-distribution with \( n-1 \) df, such as in problem 2.20. (Also see the Example 2.2 on pp. 51-52 of Montgomery’s 8th edition, and Problems 2.32, 2.33 & 2.34 all on the paired t-test).

**Statistical Inference on σ²**

We use the fact that the SMD (sampling distribution) of the random variable \( (n - 1)S^2/\sigma^2 \) from a normal (or Laplace-Gaussian) universe follows a Chi-square (\( \chi^2 \)) distribution with \( (n - 1) \) df. As an example, consider the problem 2.31 on page 63 of Montgomery’s 8th edition.

Data Statistics: \( \sum_{j=1}^{20} y_j = 116.56, \bar{y} = 5.828, \text{USS} = \sum y_j^2 = 694.3302, S^2 = 0.79045 \)

\[ \text{CSS} = 694.3302 - 116.56^2/20 = 15.01852; \text{Figure 4 atop the next page shows that} \]

\[ \chi^2_{0.95,19} = 10.1170 \rightarrow \text{Pr}(\chi^2_{19} \geq 10.1170) = 0.95 \rightarrow \text{Pr}[(n - 1)S^2/\sigma^2 \geq 10.1170] = 0.95 \rightarrow \]

\[ 0 < \sigma^2 \leq 19(0.79045)/10.1170 \rightarrow 0 < \sigma^2 \leq 1.48448 \rightarrow 0 < \sigma \leq 1.21839 \rightarrow \]

We are 95% confident that the process variance \( \sigma^2 \) lies within the interval \((0, 1.4845]\). The Pr that this last interval includes \( \sigma^2 \) is 0 or 1. The above CI implies that we cannot reject the null hypothesis \( H_0: \sigma^2 = 1.20 \) versus the alternative \( H_1: \sigma^2 < 1.20 \), i.e., we cannot conclude that \( \sigma^2 < 1.20 \) at the 5% level; however, we can reject the null hypothesis \( H_0: \sigma^2 = 1.60 \) versus the alternative \( H_1: \sigma^2 < 1.60 \) because 1.60 is outside the 95% CI \((0 < \sigma^2 \leq 1.48448]\). Note that 10.1170 represents the 95th percentage point of Chi-square, i.e., \( 10.1170 = \chi^2_{0.95,19} \), or its 0.05 quantile.
Statistical Inference on Two Normal Population Parameters

Although Montgomery covers the test of two-variance equality at the end of Chapter 2 (pp. 58-59) and a pretest on $\sigma_1^2 = \sigma_2^2$ is judicious in order to determine whether to pool the variances from 2 independent populations, we will first cover inferences about variances of two normal populations, followed by inferences on two independent population means.

Statistical Inference on Two Variances

Sir Ronald A. Fisher’s $F$ statistic describes the sampling distribution of the ratio of 2 scaled $\chi^2$ distributions give below:

$$F = \frac{\chi^2_1 / \nu_1}{\chi^2_2 / \nu_2} = \frac{[\frac{(n_1 - 1)S_1^2}{\sigma_1^2}]/(n_1 - 1)}{[\frac{(n_2 - 1)S_2^2}{\sigma_2^2}]/(n_2 - 1)} = \frac{S_1^2 / \sigma_1^2}{S_2^2 / \sigma_2^2},$$

where $\nu_1 = n_1 - 1$ is the df of the numerator and $\nu_2 = n_2 - 1$ is the df of the denominator. The Modal point = MO = 17
point of an F distribution is roughly \( \cong 1 \) for \( \nu_1 \) and \( \nu_2 > 8 \) (but always less than 1). Now, consider the

![Figure 5](https://example.com/f5.png)

Example 2.3 on pages 58-59 of Montgomery: \( n_1 = 12, S_1^2 = 14.5, n_2 = 10, \) and \( S_2^2 = 10.8 \). Figure 5 above clearly shows that the \( \Pr(F_{11,9} \leq 3.1025) = 0.95 \), i.e., the cdf of the rv \( F_{11,9} \) at \( F_{0.05,11,9} = 3.1025 \) is equal to 0.95. Put differently, the 5 percentage point of the F-distribution, or the 0.95 quantile, with \( \nu_1 = 11 \) and \( \nu_2 = 9 \) is given by \( F_{0.05,11,9} = 3.1025 \). Consequently, \( \Pr(F_{11,9} \leq 3.1025) = 0.95 \) →

\[
\Pr\left( \frac{S_1^2}{S_2^2} \leq 3.1025 \right) = 0.95 \quad \rightarrow \quad \Pr\left( \frac{S_1^2}{3.1025 S_2^2} \leq \frac{\sigma_1^2}{\sigma_2^2} < \infty \right) = 0.95 \quad \rightarrow \quad 0.4327 \leq \frac{\sigma_1^2}{\sigma_2^2} < \infty.
\]

The above CI is consistent with testing \( H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1 \) vs the alternative \( H_1: \frac{\sigma_1^2}{\sigma_2^2} > 1 \) because the CI encloses the null-hypothesized value of \( \frac{\sigma_1^2}{\sigma_2^2} = 1 \).

However, suppose we had \( n_1 = 12, S_1^2 = 14.5, n_2 = 10 \) but \( S_2^2 = 4.42 \). Now the test statistic \( F_0 = 3.2805 > F_{0.05,11,9} = 3.1025 \) leads to the rejection of \( H_0 \) at the 5% level; however, the 95% 2-sided CI: \( 0.8386 \leq \frac{\sigma_1^2}{\sigma_2^2} \leq 11.7703 \), where \( 0.8386 = \frac{S_1^2}{S_2^2} F_{0.025,11,9} = 3.9121 \), contains
\( \sigma_1^2 / \sigma_2^2 = 1 \), which is contradictory to the rejection of \( H_0 \)! The correct one-sided CI: \( 1.0574 \leq \sigma_1^2 / \sigma_2^2 < \infty \) excludes the null hypothesized value of \( \sigma_1^2 / \sigma_2^2 = 1 \) as required. Note that for the right-tailed alternative \( H_1: \sigma_1^2 / \sigma_2^2 > 1 \), we must obtain a lower one-sided CI because the upper one-sided 95\% CI: \( 0 < \sigma_1^2 / \sigma_2^2 \leq \frac{S_1^2}{S_2^2} \) excludes the null hypothesized value of \( \sigma_1^2 / \sigma_2^2 = 1 \) as required.

Testing the Equality of Two Independent Population Means

(a) Case of \( H_0: \sigma_1^2 = \sigma_2^2 = \sigma^2 \) not rejected at the 20\% level. Consider the 2-sided hypothesis \( H_0: \mu_1 - \mu_2 = \delta \) versus \( H_1: \mu_1 - \mu_2 \neq \delta \). Then, the Test Statistic is

\[ t_0 = \left[ (\bar{y}_1 - \bar{y}_2) - \delta \right] / \left( S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right), \]

where \( S_p^2 = (CSS_1 + CSS_2) / df \). Values of \( |t_0| > t_{\alpha/2, n_1 + n_2 - 2} \) lead to the rejection of \( H_0 \), i.e., the rejection region is \(( -\infty, -t_{\alpha/2, n_1 + n_2 - 2}) \cup (t_{\alpha/2, n_1 + n_2 - 2}, \infty) \). See the example on pp. 38-39 of Montgomery, and problems 2.26, 2.27. Note that a pretest on \( H_0: \sigma_1^2 = \sigma_2^2 \) at the 20\% yields \( F_0 = S_1^2 / S_2^2 = 0.10013777 / 0.06146222 = 1.6293 \), which yields a \( P\text{-value} = 0.4785 > 0.20 \), and hence the null hypothesis \( H_0: \sigma_1^2 = \sigma_2^2 \) is tenable. Thus, we may use the pooled t-statistic to test \( H_0: \mu_1 - \mu_2 = \delta \), i.e., the \( P\text{-value} \) of the pretest has exceeded 20\% providing convincing evidence in favor of pooling variances. Note that \( H_0 \) declares that \( \sigma_1^2 \) and \( \sigma_2^2 \) have a common value of \( \sigma^2 \).

Analysis of Data in Table 2.1 on page 26 of Montgomery’s 8th Edition

\( Y \) = Tension Bond Strength is measured in Kgf/cm\(^2\). Experimental Group: \( y_{ij} \): 16.85, 16.40, 17.21, 16.35, 16.52, 17.04, 16.96, 17.15, 16.59, 16.57 \( \rightarrow \bar{y}_1 = 16.7640 \); \( USS_1 = 2811.2182 \), \( CF_1 = 167.64^2 / 10 = 2810.31696 \rightarrow CSS_1 = 0.90124 \rightarrow S_1^2 = 0.10013777 \rightarrow S_1 = 0.31645 \).
Control Group: $y_2: 16.62, 16.75, 17.37, 17.12, 16.98, 16.87, 17.34, 17.02, 17.08, 17.27$

$\bar{y}_2 = 17.042, \text{ USS}_2 = 2904.85080, \text{ CF}_2 = 170.42^2/10 = 2904.2976400 \rightarrow \text{ CSS}_2 = 0.5531600 \rightarrow S_2^2 = 0.0614622$ → $S_2 = 0.24792$ (Control group), and as a result

$$S_p^2 = \frac{v_1 S_1^2 + v_2 S_2^2}{v_1 + v_2} = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}$$

Only when $n_1 = n_2 = n$, then $S_p^2 = (S_1^2 + S_2^2)/2 = 0.08080 \rightarrow S_p = \sqrt{0.0808} = 0.2842534$.

Assuming that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is not rejected at the 20% level, then $t_0 = [(\bar{y}_1 - \bar{y}_2) - \delta]/se$, where $se = se(\bar{y}_1 - \bar{y}_2) = S_p \sqrt{(1/n_1) + (1/n_2)} = 0.2842534(0.10 + 0.10)^{0.5} = 0.127122 \rightarrow t_0 = [-0.2780 - 0]/0.127122 = -2.1869; \ t_{0.025,18} = 2.100924 \rightarrow$ Reject $H_0: \mu_1 - \mu_2 = 0$ at the LOS $\alpha = 0.05$ because $|t_0| > t_{0.025,18}$. The $P-value = 2 \times \Pr(T_{18} \leq -2.1869) = 2 \times 0.021097343 = 0.042194685 < 0.05$ because $H_0$ was rejected at the 5% level. See Table 2.2 on p. 41 of Montgomery.

(2) Case of $\sigma_1^2 \neq \sigma_2^2$

Test statistics: $t_0 = [(\bar{y}_1 - \bar{y}_2) - \delta]/\sqrt{(S_1^2/n_1) + (S_2^2/n_2)}$, but the df is given by $\nu$ in equation (2.32) on page 48 of Montgomery’s 8th edition and generally $\min(n_1, n_2) < \nu \leq n_1 + n_2 - 2$. A simplified version of that Eq. (2.32) of Montgomery is given by $\nu = \frac{v_1v_2[v(\bar{y}_1) + v(\bar{y}_2)]^2}{v_2[v(\bar{y}_1)]^2 + v_1[v(\bar{y}_2)]^2}$

$$\frac{v_1v_2(F_0R_n + 1)^2}{v_2(F_0R_n)^2 + v_1},$$

where $v(\bar{y}_1) = S_1^2/n_1$, $R_n = n_2/n_1$ and $F_0 = S_1^2/S_2^2$. For equal sample sizes, the above formula reduces to $\nu = \frac{(n-1)(F_0 + 1)^2}{(F_0)^2 + 1}$. For the sake of illustration we assume that $\sigma_1^2 \neq \sigma_2^2$, so that for the data of Montgomery’s Table 2.1, the $se(\bar{y}_1 - \bar{y}_2) = [(S_1^2/n_1) + (S_2^2/n_2)]^{0.50} = 0.127122$, $t_0 = -2.1869$, and $\nu$ is also given by the formula in Table 2.4 on page 52.
of Montgomery’s 8th edition. \[ \nu = \frac{9(0.01616)^2}{(0.01001377)^2 + (0.0061422)^2} = 17.025 \]

-Value = \[2 \times \text{Pr}(T_{17.025} \leq -2.1869) = 2 \times 0.0429962\] (see Table 2.2, p.41), which is bit more conservative than that of the pooled t-test \textit{P-value}. Note that \( F_0 = 1.62926 \) and \[ \nu = \frac{9(F_0 + 1)^2}{(F_0)^2 + 1} \] leads to the same answer of \( \nu = 17.025 \). Further, Montgomery also covers the paired t-test on pp. 53-57 and Problems 2.32, 2.33 & 2.34. The pertinent hardness-example with data in Table 2.6 will be discussed in class.

\textbf{The Relative Efficiency in Hypothesis Testing}

The relative efficiency (RELEFF) of an \( \alpha \)-level statistical test \( T_1 \) to an \( \alpha \)-level test \( T_2 \) is given by \( n_2/n_1 \) iff both tests have identical values of type II error probability \( \beta \). As an example, if \( T_1 \) requires a sample of size \( n_1 = 20 \) and has \( \alpha = 0.05 \), \( \beta = 0.10 \), but \( T_2 \) requires \( n_2 = 25 \) to attain the same \( \alpha = 0.05 \) and \( \beta = 0.10 \), then the efficiency of \( T_1 \) relative to \( T_2 \) is given by \( 25/20 = 125\% \), or the RELEFF (of \( T_2 \) to \( T_1 \)) = \( 20/25 = 80\% \). On the other hand, if the 5\% level tests \( T_1 \) and \( T_2 \) both use the same random sample of size \( n = n_1 = n_2 = 25 \), but \( \beta(T_1) = 0.10 \) while \( \beta(T_2) = 0.125 \), then the RELEFF of \( T_1 \) to \( T_2 \) is given by \( 0.125/0.10 = 125\% \). Further, suppose the RELEFF (\( T_1, T_2 \)) = 125\%, both having the same \( \alpha \) & \( \beta \), and \( T_2 \) has a sample size \( n_2 = 30 \). Then the sample size for \( T_1 \) must be obtained from RELEFF (\( T_1, T_2 \)) = 1.25 = \( n_2/n_1 = 30/n_1 \rightarrow n_1 = 30/1.25 = 24 \).

\textbf{Errata for Chapter 2 of Montgomery’s 8th Edition}

1. Page 33, in Figure 2.5 change \( \sigma^2 \) to \( \sigma \).
2. Page 37, atop the page in the description of Figure 2.10, change the terminology critical region to either critical values or critical limits (or possibly rejection thresholds).
3. Page 59, the 2-sided CI on variance ratio \( \sigma_1^2 / \sigma_2^2 \) in Eq. (2.50) should appropriately be changed to \( 0.4331 \leq \sigma_1^2 / \sigma_2^2 < \infty \), because the test on \( \sigma_1^2 / \sigma_2^2 = 1 \) is right-tailed.