A FINITE ELEMENT STUDY OF ELASTO-PLASTIC HEMISPHERICAL CONTACT

Robert L. Jackson (Member, ASME)
Itzhak Green (Fellow, ASME)
George W. Woodruff School of Mechanical Engineering
Georgia Institute of Technology

ABSTRACT

This work presents a finite element study of elasto-plastic hemispherical contact. The results are normalized such that they are valid for macro contacts (e.g., rolling element bearings) and micro contacts (e.g., asperity contact). The material is modeled as elastic-perfectly plastic. The numerical results are compared to other existing models of spherical contact, including the fully plastic case (known as the Abbott and Firestone model) and the perfectly elastic case (known as the Hertz contact). At the same interference, the area of contact is shown to be larger for the elasto-plastic model than that of the elastic model. It is also shown, that at the same interference, the load carrying capacity of the elasto-plastic modeled sphere is less than that for the Hertzian solution. This work finds that the fully plastic average contact pressure, or hardness, commonly approximated to be a constant factor (about three) times the yield strength, actually varies with the deformed contact geometry, which in turn is dependant upon the material properties (e.g., yield strength). The results are fit by empirical formulations for a wide range of interferences and materials for use in other applications.

NOMENCLATURE

\[ A = \text{area of contact} \]
\[ C = \text{critical yield stress coefficient} \]
\[ E = \text{elastic modulus} \]
\[ H = \text{hardness} \]
\[ H_G = \text{hardness geometric limit} \]
\[ K = \text{hardness factor} \]
\[ P = \text{contact force} \]
\[ R = \text{radius of hemispherical asperity} \]
\[ S_y = \text{yield strength} \]
\[ a = \text{radius of the area of contact} \]
\[ e_y = \text{uniaxial yield strain, } S_y/E \]
\[ k = \text{mean contact pressure factor} \]
\[ p_o = \text{maximum contact pressure} \]

\[ z = \text{axis of symmetry for hemisphere} \]
\[ \omega = \text{interference between hemisphere and surface} \]
\[ \nu = \text{Poisson's ratio} \]

Subscripts

\[ E = \text{elastic regime} \]
\[ F = \text{fit to current FEM data} \]
\[ c = \text{critical value at onset of plastic deformation} \]
\[ o = \text{maximum} \]
\[ t = \text{transitional value from elastic to elasto-plastic behavior} \]

Superscripts

\[ = \text{equivalent} \]
\[ * = \text{dimensionless}. \]

INTRODUCTION

The modeling of elasto-plastic hemispheres in contact with a rigid surface is important in contact mechanics on both the macro and micro scales. This work presents a dimensionless model that is valid for both scales. In the former, e.g., rolling element bearings, load may be high and the deformations excessive. In the latter, e.g., asperity contact, a model on the micro-scale is of great interest to those investigating friction and wear. In addition, the real area of contact of such asperities will affect the heat and electrical conduction between surfaces. In either scale contact is often modeled as a hemisphere against a rigid flat.

One of the earliest models of elastic asperity contact is that of Greenwood and Williamson [1]. This model (GW) uses the solution of the contact of an elastic hemisphere and a rigid flat plane, otherwise known as the Hertzian solution, to stochastically model an entire contacting surface of asperities with a postulated Gaussian height distribution. The GW model also assumes that the asperities do not interfere with adjacent asperities. The Gaussian distribution is often approximated by an exponential distribution to allow for an analytical solution, although Green [2] has analytically solved the integrals using the complete Gaussian height distribution.
The Hertzian solution [3] provides closed-form expressions to the deformations and stresses of two spheres in a purely elastic contact. The two spheres may have different radii and different elastic properties. However, the closed-form solutions render an equivalent case where a single elastic sphere, having an equivalent elastic modulus, \( E' \), and an equivalent radius, \( R \), is in contact with a rigid flat (see Fig. 1, and Eqs. (1-4) that follow). The interference, \( w \), can be described as the distance from a point not deformed in the sphere to the rigid surface. The Hertz solution assumes that the interference is small enough such that the geometry does not change significantly. The solution also approximates the sphere surface as a parabolic curve with an equivalent radius of curvature at its tip. The resulting equations for contact radius and load from the Hertz solution are:

\[
wp = \frac{RA}{E'} \quad (1)
\]

\[
\frac{2}{3} E' \left( \frac{3}{4} w \right) = RE' \quad (2)
\]

\[
\frac{1}{E'} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \quad (3)
\]

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad (4)
\]

and \( E_1, \nu_1, R_1, E_2, \nu_2, R_2 \) are the elastic properties and radii of sphere 1 and 2, respectively.

Abbott and Firestone [4] stated that under fully plastic conditions the area of contact of an asperity pressed against a rigid flat can be approximately calculated by truncating the asperity tips as the rigid flat translates an interference, \( w \). For a hemisphere, this approximated fully plastic area is given by:

\[
A_{AF} = 2\pi R w \quad (5)
\]

Using Eq. (5) the contact load of the hemispherical asperity is calculated by the average contact pressure, which in this case is the hardness, since the contact is assumed to be fully plastic. The approximated fully plastic contact force is thus:

\[
P_{AF} = 2\pi R \omega H \quad (6)
\]

From this point forward, Eqs. (5) and (6) will be referred to as the AF model.

While in the elastic regime, the stresses within the hemisphere increase with \( P \) and \( \omega \). These stresses eventually cause the material within the hemisphere to yield. The interference at the initial point of yielding is known as the critical interference, \( \omega_0 \). The current work derives this critical interference analytically using the von Mises yield criterion (VM). The derivation is given in the Appendix, resulting in:

\[
\omega_0 = \left( \frac{\pi \cdot C \cdot S_y}{2E'} \right)^{2} R \quad (7a)
\]

where \( C \) is derived in the Appendix to be

\[
C = 1.295 \exp(0.736\nu) \quad (7b)
\]

The Poisson’s ratio, \( \nu \), to be used in Eq. (7b) is that of the material which yields first. For \( \nu = 0.32 \), as is used in this work, Eq. (7b) results in \( C = 1.639 \).

The critical load, \( P_c \), is then calculated from the critical interference, \( \omega_0 \), by substituting Eq. (7a) into Eq. (2). The resulting critical contact force at initial yielding is thus:

\[
P_c = \frac{4}{3} \left( \frac{R}{E'} \right)^{2} \left( \frac{C}{2} \pi \cdot S_y \right)^{3} \quad (8)
\]

Similarly, the critical contact area is calculated from Eq. (1) and is given by:

\[
A_c = \pi \left( \frac{CS_y R}{2E'} \right)^{2} \quad (9)
\]

These critical values predict analytically the onset of plasticity. These values are, therefore, chosen to normalize the results of all the models. The normalized parameters are:

\[
\omega^* = \omega / \omega_0 \quad (10a)
\]

\[
P^* = P / P_c \quad (10b)
\]

\[
A^* = A / A_c \quad (10c)
\]

Normalizing the Hertzian contact area (Eq. (1)) and force (Eq. (2)), and the AF contact area (Eq. (5)) and force (Eq. (6)), by the critical values given in Eqs. (8) and (9), results in the following simplified expressions:

\[
A^*_E = \omega^* \quad (11)
\]

\[
P^*_E = \left( \omega^* \right)^{3/2} \quad (12)
\]
\[ A_{AF}^* = 2\omega^* \quad (13) \]

\[ P_{AF}^* = \frac{3H}{CS_y} \omega^* \quad (14) \]

Chang et al. [5] (CEB model) approximated elasto-plastic contact by modeling a plastically deformed portion of a hemisphere using volume conservation. This CEB model assumes that: (1) that the hemisphere deformation is localized to near its tip, (2) the hemisphere behaves elastically below the critical interference, \( \omega_c \), and fully plastically above that value, and (3) the volume of the plastically deformed hemisphere is conserved. Also, the critical interference used in the CEB model is given by:

\[ \omega_c = \left( \frac{\pi KH}{2E'} \right)^2 R \quad (16) \]

where \( K \) is the hardness factor given by \( K = 0.454 + 0.41v \) and the hardness is assumed \( H = 2.8S_y \). While from an engineering perspective the corresponding values given by Eq. (7) and Eq. (16) are very close, the CEB model is limited to this fixed relationship between the hardness and the yield strength. It should be noted that Eq. (7) is not limited by any such assumption. Likewise the CEB model contains a discontinuity at \( \omega_c \).

Zhao et. al. [6] devised an elasto-plastic (ZMC) model, which interpolates between the elastic and fully plastic (AF) models. The ZMC model divides the interference into three segments: (1) elastic (Hertz), (2) elasto-plastic (using a template) and (3) fully plastic (AF). A template function satisfies continuity of the function and its slope at the two transitions.

The most widely used prediction for the hardness is Tabor’s [7], who calculates it to be approximately three times the yield strength for most cases. More recently, Mesarovic and Fleck [8] studied the contact of elastic-perfectly plastic spheres using the finite element analysis. They plot \( P/(A_s) \) as a function of \( a/R \), i.e., the ratio of the average contact pressure to the yield strength versus the radius of the contact (or equivalently the deformation). An interesting trend then emerges. The average pressure appears to have a limiting function, now designated by \( H_C \). The limiting function, \( H_C \), changes from a value of approximately \( 3S_y \) (Tabor’s value) to significantly lower values with an increasing ratio of \( a/R \). Thus, the hardness is found to change in relation to the contact geometry after deformation. The current work produces precisely the same trend.

Kogut and Etsion [9] also performed a FEM analysis of the same case of an elastic-perfectly plastic sphere in contact with a rigid flat. Again in their analysis, the value of \( H \) is set to be fixed at 2.8\( S_y \). Notably, the slope of \( P/(A_s) \) is not zero (it still increases monotonically) at the point where full plasticity is assumed. Their work gives a very detailed analysis of the stress distribution in the contact region, and empirical expressions are provided for the contact area, the contact force, and the average contact pressure. The resulting equations have a discontinuous slope at \( \omega = 6 \), and they describe the deformation only up to \( \omega = 110 \), at which point full plasticity is assumed.

The current work uses the finite element method to model the case of an elastic-perfectly plastic sphere in contact with a rigid flat. The von Mises criterion defines the yielding of the material. The resulting numerical data is also fitted by continuous functions that capture deformations all the way from purely elastic to fully plastic conditions. Theses expressions, that have a relatively low statistical error, may be used in other applications whether they are on macro or micro scales. For example, a statistical model for asperity contact (such as GW [11]) can greatly benefit from such expressions.

**FINITE ELEMENT MODEL**

To improve upon the efficiency of computation, an axisymmetric 2-D model is used. The present study utilizes the commercial program ANSYS™, while ABAQUS™ produces the same results. Kogut and Etsion [9] also use ANSYS™. However, the mesh in the current analysis is orders of magnitude more refined as necessitated by mesh convergence.

The contact region is meshed by 100 contact elements. These are in essence very stiff springs attached between surface nodes, and activate only when penetration onset into the rigid flat is detected. This contact region varies in order to fit the expected area of contact. The contact elements thus apply forces to the nodes of the elements that are in contact.

![Finite element mesh of sphere generated by ANSYS™](image)

The model refines the element mesh near the region of contact to allow the hemisphere’s curvature to be captured and accurately simulated during deformation. The model uses quadrilateral, four node elements to mesh the hemisphere, but the results have also been confirmed using a mesh of eight node elements. The resulting ANSYS™ mesh is presented in Fig. 2, where ABAQUS™ produces a similar mesh. The quarter-circle mesh represents the axisymmetric hemispherical body, and the straight line represents the rigid plane.

The contact force acting on the hemisphere is found from the reaction forces on the hemisphere base nodes that retain the desired interference. The radius of the contact area is
In order to verify the validity of the model, mesh convergence must be satisfied. The mesh density was iteratively increased by a factor of two until the contact force and contact area differed by less than one percent between iterations. The resulting mesh consists of at least 11,101 elements, since the number of meshed elements will vary with the expected region of contact. The stiffness of the contact elements were also increased by an order of magnitude in successive iterations until the solution differed by no more than one percent between successive iterations.

In addition to mesh convergence, the model also compares well with the Hertz elastic solution at interferences below the critical interference. The contact force of the model differs from the Hertzian solution by no more than two percent. The contact radius differs by a maximum of 8.1%, but the average error is only 4.4%. When the contact areas are calculated from the radii, the maximum error increases to 17%. The smaller error in the contact force is attributed to overall force balance (static equilibrium) enforced by the FEM packages. However, the contact radius is obtained from a discrete mesh (which has a finite resolution). Moreover, the magnitude of the contact element stiffness also has some effect upon such radii, but not on the overall force balance. Generally, though, the differences are small enough that the FEM solution practically conforms to the Hertzian solution at interferences below critical (and even slightly above).

There are two ways to simulate the contact problem. The first applies a force to the rigid body and then computes the resulting displacement. The second applies a displacement and then computes the resulting contact force. In both methods, the displacement, stress, and strain in the elastic body can be determined, as well as the contact pressure. In this model the latter approach is used, where the base nodes of the hemisphere are displaced an interference, \( \omega \) approaching the rigid flat surface. The radial displacements of the base nodes are restricted. This method is used because the resulting solution converges more rapidly than the former.

The contact problem and the elasto-plastic material property make the analysis highly non-linear and difficult to converge. An iterative scheme is used to solve for the solution and many load steps are used to enhance solution convergence. Initially, a small interference is set of the total interference and then it is incremented after the load step converges. This continues until a converged solution is found for the desired interference.

**NUMERICAL RESULTS AND DISCUSSION**

The results of the described finite element hemisphere model are presented for a variety of interferences. The yield strengths cover a typical range of steel materials used in engineering [10]. While the elastic modulus and Poisson’s ratio are held constant at 200 GPa and 0.32, respectively, five different material yield strengths are modeled. These are designated Mat.1 through Mat.5 corresponding to their yield strengths which are 0.210 GPa, 0.5608 GPa, 0.9115 GPa, 1.2653 GPa, and 1.619 GPa. Once the mesh is generated, computation takes from two minutes for small interferences to about one hour for large interferences on a 2.5 GHz PC.

The dimensionless contact area is normalized by the Hertz solution (Eq. (11)) and plotted as functions of \( \omega^* \) in Fig. 3. The data is presented on a log scale to capture the entire range of interferences. While \( \omega < 1.9 \) the finite element model agrees well with the Hertz solution (\( A^\prime/\omega^* = 1 \)). This is likely because the plastic deformations are still relatively small such that the Hertz solution is not dramatically affected. As the sphere begins to plastically deform below the surface, the sphere weakens and thus does not retain its shape as well as if it were perfectly elastic throughout. Thus the area of contact eventually becomes larger in the elasto-plastic case than in the perfectly elastic case. The FEM model values for contact area continue to increase with interference even past Abbot and Firestone’s fully plastic (AF) model [4] at \( A^\prime/\omega^* = 2 \). Since the AF model is based on the truncation of the contacting geometries, it does not model the actual deformation of the hemisphere. It seems reasonable then that the FEM solution for contact area could continue past the AF model.

![Figure 3: FEM predicted contact area.](image)

Overall though, the FEM model generally follows the Hertz elastic solution near the critical interference and then increases past the AF model as the interference increases. Later in this work, this trend will be followed by empirical formulations fitted to the data. The FEM results also indicate a material dependence of the normalized contact area. Since the contact area is calculated by counting the number of elements in contact, and there are only a finite number of such elements, there is an inherent error in the data. The scatter in the data can be attributed mostly to this, and to the fact that the FEM is yet a discrete formulation.

For the contact area, all the models follow the same trend but differ in magnitude. The ZMC model follows the Hertz elastic solution at low and moderate interferences, but abruptly migrates to the AF model before the current model and the KE model. The KE model and the current empirical model also...
agree fairly well on average, except at large interferences. The KE model clearly shows a slight discontinuity at $\omega^* = 6$ and then terminates at $\omega^* = 110$. The KE model does not connect with the Hertz elastic solution at the critical interference depth. Also, the nondimensional KE model is material independent such that its contact area falls between the data of materials 1 and 5 of this work.

The dimensionless contact force is normalized by the Hertz solution (Eq. (12)) and plotted as a function of $\omega^*$ in Fig. 4. This plot uses a log scale to capture the entire range of interferences. The normalized contact force $(P^*/(\omega^*)^{3/2})$ calculated from the current model follows precisely the Hertz elastic solution $(P^*/(\omega^*)^{3/2}) = 1$ at small interferences. With increasing interference the current model eventually increases toward the AF model [4]. It is interesting to note that the AF model predicts higher loads at small interferences than the Hertzian solution, but eventually crosses over to become the lesser of the two. This is because the AF model assumes a constant pressure distribution, which is equal to the hardness, while the average pressure of the Hertzian solution is initially lower than the hardness. At higher interferences, the FEM data displays a material dependent behavior.

The trends of all the models are similar; however, the ZMC again crosses to the AF model prematurely. At low interferences, the KE and ZMC models predict a contact force that is greater than the elastic model. This cannot be the case, as the yield strength of the material limits the stiffness of the hemisphere. Again the KE model shows a discontinuity at $\omega^* = 6$ and then terminates at $\omega^* = 110$. Generally the KE model and the current FEM results are very similar. At about $\omega^* = 50$ the KE model crosses over the current model and continues to overestimate the contact force until $\omega^* = 110$. The KE and ZMC models also fail to capture the material dependence effects at large interferences.

The average contact pressure to yield strength ratio, $P/(A_S)_y$, is calculated from the data and plotted in Fig. 5 alongside the hertz contact solution. The Firestone and Abbott [4] fully plastic (AF) model is represented by the horizontal line at $P/(A_S)_y = 3$. The average contact pressure should approach the hardness of the material as the contact becomes fully plastic. It is widely accepted that the hardness is approximated by $3(S_y)$ [7]. It becomes evident in this plot that this is not always the case. From the data it seems that hardness is not a constant material property. The cause of this trend will be discussed later in greater detail. The work by Mesarovic and Fleck [8] also confirms this trend.

**Figure 4: FEM predicted contact force.**

**Figure 5: Average contact pressure to yield strength ratio.**

**EMPIRICAL FORMULATION**

General empirical approximations of the FEM data are desired which can be used at any deflection and for any set of material properties. This will help designers in a variety of single contact problems, as well as it can be readily incorporated into statistical models to model rough surfaces.

As mentioned previously, the FEM solution for the area of contact continues past the AF model with increasing interference. Hence, the leading coefficient in Eq. (13) is allowed to vary when equations are fitted to the FEM data. This is reasonable, since the AF model is not an exact solution (it is based on a truncation assumption). Here a power function is used in place of this leading coefficient and fit to the numerical data. Figures 3-5 show that there are two distinct regions in the FEM data, thus a piecewise formulation is used to fit the data. At small interferences the Hertz solution is assumed and at large interferences the power function is fit to the FEM data, resulting in:

For $0 \leq \omega^* \leq \omega_i^*$

$$A_F^* = \omega^*$$  \hspace{1cm} (17a)

For $\omega_i^* \leq \omega^*$

$$A_F^* = \omega^* \left( \frac{\omega^*}{\omega_i^*} \right)^B$$  \hspace{1cm} (17b)

where
The value $\omega_i^*$ represents the transition point from elastically dominant behavior to elasto-plastic behavior. The formulation follows the Hertzian solution (Eq. (17a)) for $\omega < 1.9$. Then it transitions to the elasto-plastic case and eventually continues past the AF model for high values of $\omega$. Equation (17b) is also somewhat dependant on the material properties, according to the definition in Eqs. (17c-d). Statistically, Eq. (17) differs from the FEM data for all five materials by an average of 1.3% and a maximum of 4.3%. An equation of the same form as the ZMC model fitted to the FEM data results in an average error of 43.2%. Notably, Eqs. (17a) and (17b) are continuous at $\omega_i^*$.

A Weibull function fitted to the limiting values of $H_G/S_y$ yields:

$$
\frac{H_G}{S_y} = 2.84 \left[ 1 - \exp \left( -0.82 \left( \frac{a}{R} \right)^{-0.7} \right) \right]
$$

(19)

This substitution is valid only when $\omega^* \geq \omega_i^*$. Equation (19) can then be substituted into Eq. (18) so it may then be rewritten as a function of $\omega^*$ as follows:

$$
\frac{H_G}{S_y} = 2.84 \left[ 1 - \exp \left( -0.82 \left[ \frac{\pi C e_y}{2 \sqrt{\omega^* \omega_i^*}} \right]^{2/3} \right) \right]
$$

(20)

This results in a formulation for $H_G$ as a function of the material properties, $E$, $S_y$, and $v$ (not just upon $S_y$ as suggested by Tabor [7]). Interestingly, as $a/R$ approaches zero, the limiting value of $H_G/S_y=2.84$ agrees almost precisely with the theoretical value of 2.83 (Williams [11, p.109]).

To formulate an approximation of the contact force as predicted by the FEM results, the AF model for contact force must first be corrected by way of substituting in Eq. (18) or Eq. (20) into Eq. (14), letting $H_G$ replace $H$, and by allowing the AF contact area to deviate from Eq. (13) (see reasoning for Eq. (17)). This results in an equation for a corrected fully plastic model. Once again a piecewise solution is fit to the FEM data. At small interferences, the Hertz solution is assumed. The resulting piecewise equation fit to the FEM data is given as:

$$
P^* = \left( \omega^* \right)^{1/2}
$$

(21a)

In order to formulate a fit for the FEM contact force, the material dependent trend at high interferences shown in Fig. 4 is modeled. To assist in this model, a plot of $P(A/S_y)$ as a function of $a/R$ in Fig. 6 reveals the cause of the material dependency. In this plot a limit appears to emerge for the fully plastic average pressure, commonly referred to as the hardness. Here the hardness appears to change as a function of $a/R$, or with the evolving geometry of contact. The trend may be explained by the progression schematically shown in Fig. 7. As the interference increases and the contact geometry changes, the limiting average pressure to yield strength ratio, $H_G/S_y$, must change from Tabor's predicted value of three to a theoretical value of one when $a=R$. The contact region when $a=R$ is essentially the case of a deformable blunt rod in contact with a rigid flat whose $H_G/S_y$ value is theoretically one.
For \( \omega^* \leq \omega^* \),

\[
P_F^* = \left[ \exp \left( -\frac{1}{4} \left( \omega^* \right)^{\frac{5}{12}} \right) \right] \left( \omega^* \right)^{3/2} + \frac{4H_G}{CS_y} \left[ 1 - \exp \left( -\frac{1}{25} \left( \omega^* \right)^{\frac{5}{2}} \right) \right] \omega^* \]  

(21b)

where \( \omega^* = 1.9 \). This formulation approaches asymptotically the Hertzian elastic model at small interferences, and approaches and continues past the AF model at large interferences. Statistically this formulation differs from the FEM data for all five materials by an average error of 0.94% and a maximum of 3.5% when Eq. (20) is used for \( H_G \).

The average pressure to yield strength ratio, \( P/(A_S) \), can now be modeled by combining Eq. (17) and Eq. (21). Since these equations are normalized by their critical values, the resulting formulation for the average pressure is:

\[
\frac{P}{A \cdot S_y} = \frac{2}{3} \frac{P_F^*}{A_F^*} \]  

(22)

This ratio is shown in Fig. 8 (only the weakest and strongest materials are plotted for clarity). The largest differences between the ZMC and KE models and the current FEM model then appear. It is apparent that the KE and ZMC models do not account for material dependence in the limiting average pressure to yield strength ratio, \( H_G/S_y \). Both ZMC and KE are monotonically increasing and truncated at some point that traditionally is considered to be the “hardness.” The ZMC and KE models both estimate the average pressure in the transition from the elastic to the elasto-plastic regime fairly well. It is also apparent that these models do not intersect with the Hertzian solution at \( P/(A_S) = 2C/3 \). The discontinuity in the slope in the KE model at a value of 6 and in the current model at a value of 1.9 is also clearly evident (see Eqs. (17, 21)).

**CONCLUSIONS**

This work presents a 2D axisymmetric finite element model of an elastic-perfectly plastic hemisphere in contact with a rigid flat surface. A comparison is also made with other existing models. The material is modeled as elastic-perfectly plastic, and yielding occurs according to the von Mises criterion. A concise form is presented for the critical interference at which plastic deformation initiates within the hemisphere. It is derived from the Hertzian solution and the von Mises yield criterion. An a priori definition of the hardness is not needed.

The resulting plots indicate that the FEM results for the contact area agree closely at small interferences with the trends of the Hertzian solution. While at large interferences the FEM predicts contact areas that surpass Abbot and Firestone’s fully plastic model [4] (that is based upon truncation). The ZMC model is found to differ significantly from the FEM results, where the KE model (which is also a FEM) follows more closely and has the same general trends. An empirical formulation for the contact area is also fitted to the FEM data as a function of the material properties and interference.

The FEM results of the contact force predict a lower load carrying capacity than the AF model for most materials and values of \( \omega^* \). This is because the AF model assumes that the average pressure distribution is simply the hardness, which is approximated by \( 3S_y \). It is found, however, that the fully plastic average contact pressure or hardness is not constant as is widely accepted. Rather, the limiting value of the fully plastic average pressure varies with the deformed contact geometry, which in turn is coupled to the material yield strength. This is accounted for in an empirical formulation for the limiting average pressure to yield strength ratio, \( H_G/S_y \). A formulation using \( H_G/S_y \) is then fit to the FEM contact force data.

This work reveals large differences between approximate analytical models and other numerical solutions. More importantly, the contact area, force, and pressure that are found to be particularly dependent upon the deformed geometry in all regimes and effectively dependent upon the material properties (e.g., strength) in the elasto-plastic and plastic regimes. The fit-their-all equations that solely depend upon deformation, which are found in previous works, are imprecise when compared to current FEM results. For example, the average contact pressure to yield strength ratio in all previous work is shown to increase monotonically with deformation, and is assumed to terminate (or truncate) at the hardness. In this work it is shown that such a truncation is not warranted. Particularly, it is shown that the truncation model of Abbott and Firestone [4] cannot be justified. This work discovered significant geometrical and material nonlinearities, and that the hardness depends not just upon strength but also upon the modulus of elasticity, Poisson’s ratio, and most importantly upon the deformation itself (i.e., hardness is not a unique or fixed material property as indicated by Tabor [7], and assumed by others after him). The results are based on the finest and adaptive mesh yet (over 11,000 elements for a single hemispherical asperity in contact with a rigid flat, and 100 contact elements) that was necessary for finite element convergence. The results were obtained by using...
ANSYSTM and then independently confirmed by using ABAQUSSTM.

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APPENDIX: CRITICAL INTERFERENCE

The Hertz solution results in the following equations for stress within the deformed sphere along the axis of revolution, z (Johnson, [12]):

\[
\sigma_1 = -p_o \left( 1 + \left( \frac{z}{a} \right)^2 \right)^{-1}
\]

(A1)

\[
\sigma_{2,3} = p_o \left[ 2 \left( 1 + \left( \frac{z}{a} \right)^2 \right) \right]^{-1} \left[ 1 + \nu \left( 1 - \frac{z}{a} \tan^{-1} \left( \frac{a}{z} \right) \right) \right] \]

(A2)

where the origin of the z-axis lies at the point of initial contact between the hemisphere and the rigid flat, and \( p_o \) is the maximum contact pressure.

The von Mises yield criterion is given as:

\[
S_y = \sqrt{\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]}
\]

(A3)

By substituting the principal stresses given in Eq. (A1) and Eq. (A2) into Eq. (A3) and then simplifying, the following equation for the von Mises yield criterion is obtained:

\[
\frac{S_y}{p_o} = \frac{3}{2} \left[ \left( 1 + \left( \frac{z}{a} \right)^2 \right)^{-1} - (1+\nu) \left( 1 - \frac{z}{a} \tan^{-1} \left( \frac{a}{z} \right) \right) \right]
\]

(A4)

The resulting Eq. (A4), which must be positive, dictates where within the hemisphere initial yielding occurs. This is obtained by setting the derivative with respect to z to zero. Hence,

\[
\frac{d}{dz} \left( \frac{S_y}{p_o} \right) = -a [a^2(4+\nu) + (1+\nu)z^2] + (1+\nu)(a^2+z^2) \tan^{-1} \left( \frac{a}{z} \right) = 0
\]

(A5)

This equation is solved numerically for Poisson ratios between 0.01 and 0.50 to find the locations, z, at initial yielding. These locations are then substituted in Eq. (A4) to find the applied maximum contact pressure to yield strength ratio, \( p_o/S_y \). This ratio, \( p_o/S_y \), is referred to as the yield strength coefficient and designated by the symbol C. An empirical function is fitted to the final numerical data, which is given by:

\[
\frac{p_o}{S_y} = C = 1.295 \exp(0.736\nu)
\]

(A6)

Equation (A6) differs from the numerical solution by an average of 1.2% and by no more than 3.1%.

The interference, \( \omega \) is given as a function of \( p_o \) by the Hertz elastic solution in Johnson [12] as:

\[
\omega = \left( \frac{\pi \cdot p_o}{2E'} \right)^{2/3}
\]

(A7)

Thus, to find the critical interference, or the interference at the initial point of yielding, the maximum pressure when yielding first occurs, \( p_o=0.35H \), and is substituted into Eq. (A7) for \( p_o \). This maximum pressure is the pressure given by the von Mises contact to yield strength ratio in Eq. (A6). The equation \( p_o/S_y \) is substituted into Eq. (A7), resulting in the Eq. (7a).

A similar derivation is also given in Chang [13]. However, that derivation assumed a fixed value between strength and hardness, \( S_y=0.35H \), which resulted in an equation for the hardness coefficient, \( K = 0.454 + 0.41\nu \). Such an assumption is not made in this work (see discussion on \( H_G \) within).

REFERENCES