Distance Distribution of Bivariate Poisson Network Nodes

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***Abstract*—In this paper, we obtain the distribution of the distances between an arbitrary point and its dissimilar neighboring points in a spatial bivariate Poisson point process. We derive closed-form solutions for the real moments of the inter-nodal distances for two special cases in which there is either strong or weak correlation between the number of points in a region. This point process can be used to model heterogeneous random networks. In particular, we discuss some applications of the obtained results in cognitive wireless networks.**

***Index Terms*—Bivariate Poisson, spatial point processes, internodal distances, random networks, interference.**

I. INTRODUCTION

In the analysis of networks with randomly distributed nodes, there is a need to have the statistical distribution of distance between nodes. Applications include: signal and interference characterization and obtaining fundamental communication limits, routing and localization algorithms, spatial databases [1], etc. The recent work in distance characterization focuses on networks of homogeneous type in which nodes have no qualitative distinction (e.g., [2]). In this paper, we consider the case where each node belongs to one of the two distinguishable types (e.g., in networks with heterogeneous node types). We consider the general case where the number of nodes of each type in a given region are marginally Poisson distributed and correlated. We find the distribution of distances between an arbitrary point and its dissimilar neighboring points. We also obtain real moments of these distances for two cases of weak and strong correlation.

In wireless networks, interference is influenced by the distances between nodes. We discuss the applications of our results in interference modeling and avoidance in cognitive random wireless networks, which is an example of heterogeneous random networks.

II. SPATIAL BIVARIATE POISSON PROCESSES

We consider a spatial point process, defined on the Euclidean plane, $\mathbb{R}^2$, in which there are two types of points and each point is tagged with one of the two possible labels. We distinguish between these points as type-A and type-B. The number of type-A and type-B points in an area are each marginally Poisson distributed and also correlated. These processes are called bivariate or two-type spatial point processes [3]. There has been limited work using spatial bivariate Poisson distribution (e.g., [3] and [4]).

We consider three possible events: A, B, and AB (implying joint occurrence of both types), with the intensities λ, μ and ν, respectively. The occurrence of both types A and B at the same spatial point is somewhat of a limited interest from a practical point of view[1]. Following the approach in [4], we consider a reference spatial Poisson process with intensity $\lambda+\mu+\nu$. Given that an event of this process has occurred at the spatial point $\mathbf{y}$, with probability $\frac{\lambda}{\lambda+\mu+\nu}$, there is a type-A point at $\mathbf{y}$, with probability $\frac{\mu}{\lambda+\mu+\nu}$, there is a type-B point at $\mathbf{y}$ and with probability $\frac{\nu}{\lambda+\mu+\nu}$ there is a type-A point at $\mathbf{x_1}$ and a type-B point at $\mathbf{x_2}$ such that $\|\mathbf{x_1}-\mathbf{y}\|$ and $\|\mathbf{x_2}-\mathbf{y}\|$ are independent random variables with probability density functions $f(\|\mathbf{x_1}-\mathbf{y}\|)$ and $g(\|\mathbf{x_2}-\mathbf{y}\|)$, respectively[2].

We classify a point as either single (corresponding to events A and B) or paired (i.e., being paired with a point of the other type, corresponding to event AB). Considering $d\mathbf{x}$ as a differential area at point $\mathbf{x}$, probability that a type-A (type-B) single point exists at $\mathbf{x}$ is $\lambda d\mathbf{x}$ ($\mu d\mathbf{x}$).

Given that an event of the reference point process corresponding to paired points has occurred at $\mathbf{y}$ (this happens with probability $\nu d\mathbf{y}$), the probability that A occurs at $d\mathbf{x}_1$ and B at $d\mathbf{x}_2$ is $f(\|\mathbf{x}_1-\mathbf{y}\|)g(\|\mathbf{x}_2-\mathbf{y}\|)d\mathbf{x}_1 d\mathbf{x}_2$. To find the probability of paired points at A and B we integrate the above probability over all points in the Euclidean plane:

$$\begin{aligned}&\Pr\{\text{paired points with A at } d\mathbf{x_1} \text{ and B at } d\mathbf{x_2}\}\\ &= \nu h(\mathbf{x_1},\mathbf{x_2})d\mathbf{x_1}d\mathbf{x_2},\end{aligned} \tag{1}$$

where $h(\mathbf{x_1},\mathbf{x_2})$ is defined as

$$h(\mathbf{x_1},\mathbf{x_2}) \triangleq \int_{\mathbf{y}} f(\|\mathbf{x_1}-\mathbf{y}\|)g(\|\mathbf{x_2}-\mathbf{y}\|)d\mathbf{y}. \tag{2}$$

Since the integration of $h(\mathbf{x_1},\mathbf{x_2})$ on either $\mathbf{x_1}$ or $\mathbf{x_2}$ equals one, the probability of a type-A or type-B paired point at $d\mathbf{x}$ is $\nu d\mathbf{x}$. Consequently,

$$\Pr\{\text{a type-A point at } d\mathbf{x_1}\} = \lambda d\mathbf{x_1} + \nu d\mathbf{x_1} = (\lambda+\nu)dx_1.$$

Similarly, $\Pr\{\text{a type-B point at } d\mathbf{x_2}\} = (\mu+\nu)d\mathbf{x_2}$. So, the number of type-A points, $N(A)$, and type-B points, $N(B)$, in the two-dimensional region $\mathcal{R}$ with area S are each marginally Poisson distributed with parameters $(\lambda+\nu)S$ and $(\mu+\nu)S$, respectively and their covariance is (see e.g., [4]),

$$\text{Cov}\{N(A),N(B)\} = \nu \int_{\mathbf{x_1},\mathbf{x_2}\in\mathcal{R}} h(\mathbf{x_1},\mathbf{x_2})\ d\mathbf{x_1}d\mathbf{x_2}. \tag{3}$$

The covariance, therefore, depends on the parameter ν as well as the behavior of the function $h(\mathbf{x_1},\mathbf{x_2})$ in the region $\mathcal{R}$.

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[1] This is unlike the one-dimensional bivariate Poisson case, in which joint occurrence implies that the two events occur at exactly the same time instant.

[2] Note that $\|\mathbf{x}-\mathbf{y}\|$ in $f(\|\mathbf{x}-\mathbf{y}\|)$ is an auxiliary variable and shows a value that the random variable $\|\mathbf{x}-\mathbf{y}\|$ can take.

A. Distance to nth Nearest Neighbor Distribution

Let us denote the distance of a type-A (B) point to its nth nearest type-B (A) neighbor as $R_{AB,n}$ ($R_{BA,n}$) and define $\mathcal{E}_1$ and $\mathcal{E}_2$ as the events that there exists a type-A point at $d\mathbf{x}_1$ and there exists a type-B point at $d\mathbf{x}_2$, respectively. A type-A point at $d\mathbf{x}_1$ and a type-B point at $d\mathbf{x}_2$ can be:

- single points with probability $\lambda d\mathbf{x}_1 \mu d\mathbf{x}_2$,
- together as paired points with probability $\nu h(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2$,
- A single but B paired with probability $\lambda d\mathbf{x}_1 \nu d\mathbf{x}_2$,
- A paired but B single with probability $\nu d\mathbf{x}_1 \mu d\mathbf{x}_2$.

Then, we have:

$$\begin{aligned}
\Pr\{\mathcal{E}_2 | \mathcal{E}_1\} &= \frac{\Pr\{\mathcal{E}_1 \bigcap \mathcal{E}_2\}}{\Pr\{\mathcal{E}_1\}} \\
&= \frac{\lambda d\mathbf{x}_1 \mu d\mathbf{x}_2 + \nu h(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 + \lambda d\mathbf{x}_1 \nu d\mathbf{x}_2 + \nu d\mathbf{x}_1 \mu d\mathbf{x}_2}{(\lambda + \nu) d\mathbf{x}_1} \\
&= [\mu + \frac{\nu}{\lambda + \nu}(\lambda + h(\mathbf{x}_1, \mathbf{x}_2))] d\mathbf{x}_2. \qquad (4)
\end{aligned}$$

From (4), we see that given a type-A point exists at $d\mathbf{x_1}$, the conditional intensity of a type-B point at $\mathbf{x_2}$ will be

$$\Lambda(\mathbf{x_2}) = \mu + \frac{\nu}{\lambda + \nu}(\lambda + h(\mathbf{x_1}, \mathbf{x_2})). \qquad (5)$$

We can see that the intensity of type-B points is nonhomogeneous due to the $h(\mathbf{x_1}, \mathbf{x_2})$ term.

Let us denote the circle centered at the type-A point (i.e., at $\mathbf{x_1}$) and radius r as $\mathcal{C}$. The expected number of type-B points in $\mathcal{C}$ is:

$$\chi(r) = \int_{\mathbf{x_2} \in \mathcal{C}} \Lambda(\mathbf{x_2}) d\mathbf{x_2}. \qquad (6)$$

Lemma 1. *The pdf of the distance of a type-A point to its nth nearest type-B neighbor is*

$$f_{R_{AB,n}}(r) = e^{-\chi(r)} \frac{d\chi(r)}{dr} \frac{\chi(r)^{n-1}}{(n-1)!}. \qquad (7)$$

Proof: The Complementary Cumulative Distribution Function (CCDF) of $R_{AB,n}$ can be calculated as

$$\begin{aligned}
F_{c,R_{AB,n}}(r) &= \Pr\{R_{AB,n} > r\} \\
&= \Pr\{\text{There are less than } n \text{ type-B points in } \mathcal{C}\} \\
&= \sum_{k=0}^{n-1} e^{-\chi(r)} \frac{\chi(r)^k}{k!}. \qquad (8)
\end{aligned}$$

The pdf of $R_{AB,n}$ can be found as

$$f_{R_{AB,n}}(r) = -\frac{dF_{c,R_{AB,n}}(r)}{dr} = e^{-\chi(r)} \frac{d\chi(r)}{dr} \frac{\chi(r)^{n-1}}{(n-1)!}. \qquad (9)$$

■

Good choices for $f(\mathbf{x})$ and $g(\mathbf{x})$ are zero-mean isotropic Gaussian distributions with variances σ_1^2 and σ_2^2, respectively. Using these functions, $h(\mathbf{x}_1, \mathbf{x}_2)$ can be easily calculated as

$$h(\mathbf{x}_1, \mathbf{x}_2) = h(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{0}) = h(\mathbf{x}, \mathbf{0}) = \frac{1}{2\pi\sigma^2} e^{\frac{-\|\mathbf{x}\|^2}{2\sigma^2}}, \qquad (10)$$

where $\mathbf{x} = \mathbf{x_1} - \mathbf{x_2}$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. In this case, the probability of paired points in (1) depends only on the distance between the two points. Moreover, the correlation factor can be captured by a single parameter, i.e., σ^2.

For $h(\mathbf{x_1}, \mathbf{x_2})$ given in (10), $\chi(r)$ can be found using (5) and (6) as

$$\begin{aligned}
\chi(r) &= (\mu + \frac{\nu\lambda}{\nu + \lambda})\pi r^2 + \frac{\nu}{\nu + \lambda}(1 - e^{\frac{-r^2}{2\sigma^2}}) \\
&= ar^2 - be^{-cr^2} + b, \qquad (11)
\end{aligned}$$

where $a = (\mu + \frac{\nu\lambda}{\nu+\lambda})\pi$, $b = \frac{\nu}{\lambda+\nu}$, and $c = \frac{1}{2\sigma^2}$.

In the absence of correlation and when $\nu = 0$, we have $\chi(r) = \mu\pi r^2$ and $f_{R_{AB,n}}$ will reduce to the generalized Gamma distribution obtained in [2] . Using (3) and (10), small ν and large σ^2 imply that the correlation between type-A and type-B points is weak. In this case, we have $\sigma^2 \gg 1$ and $c \ll 1$. Therefore, we have $e^{-cr^2} \approx 1 - cr^2$ and $\chi(r) \approx (a + bc)r^2$. On the other hand, for large ν and small σ^2, the correlation is significant. In this case, we have $\sigma^2 \ll 1$ and $c \gg 1$. Therefore, we can write $\chi(r) \approx ar^2 + b$.

Lemma 2. *For α a real number and for the weak correlation case*

$$E\{R^{\alpha}_{AB,n}\} = \frac{\Gamma(n + \alpha/2)}{\sqrt{(a + bc)^{\alpha}}(n-1)!}, \ n > -\alpha/2 \qquad (12)$$

and for the strong correlation case

$$E\{R^{\alpha}_{AB,n}\} \approx \frac{e^{-b} \sum_{k=1}^{M} w_k x_k^{\alpha/2} (b + x_k)^{n-1}}{\sqrt{a^{\alpha}}(n-1)!} \qquad (13)$$

where M is a large integer and w_k and x_k are the weights and abscissas of Gauss-Laguerre quadrature of order M respectively.

Proof: Expected value of $R^{\alpha}_{AB,n}$ is found using (7) as

$$E\{R^{\alpha}_{AB,n}\} = \int_0^{\infty} r^{\alpha} e^{-\chi(r)} \frac{\chi(r)^{n-1}}{(n-1)!} d\chi(r). \qquad (14)$$

For weak correlation, using integration by substitution

$$E\{R^{\alpha}_{AB,n}\} = \frac{1}{\sqrt{(a + bc)^{\alpha}}(n-1)!} \int_0^{\infty} u^{n+\alpha/2-1} e^{-u} du, \qquad (15)$$

and (12) is found directly from the definition of Gamma function. For the case of strong correlation, $\chi(r) \approx ar^2 + b$, and after some mathematical manipulation

$$E\{R^{\alpha}_{AB,n}\} = \frac{e^{-b}}{\sqrt{a^{\alpha}}(n-1)!} \int_0^{\infty} x^{\alpha/2} (b + x)^{n-1} e^{-x} dx \qquad (16)$$

Using the Gauss-Laguerre quadrature method [6], the result in (13) is found. ■

To verify the accuracy of the above analytical results, we have considered a bivariate Poisson point process with parameters $\lambda = 1$ and $\mu = 2$ and α is assumed 4. For weak and strong correlation, we have set $\nu = 0.1$, $\sigma^2 = 10$ and $\nu = 10$, $\sigma^2 = 0.1$ respectively. In Figure 1, simulation results is shown to follow the analytical values very closely.

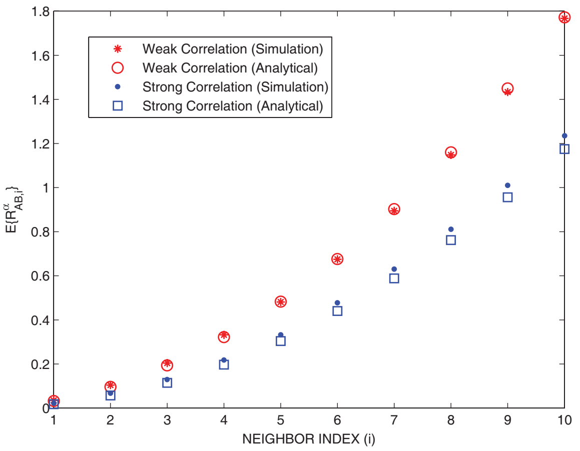

Fig. 1. Analytical and simulation results for $E\{R^{\alpha}_{AB,n}\}$.

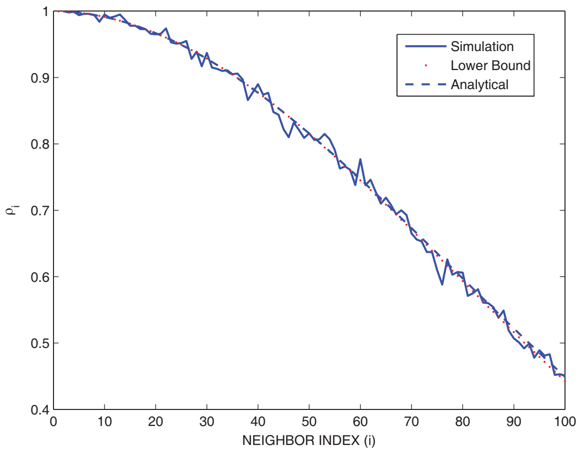

Fig. 2. Probability of abstention vs. neighbor index.

III. APPLICATIONS

Spatial bivariate Poisson process can be used in modeling heterogeneous random networks consisting of two types of nodes. One example is cognitive wireless networks which are comprised of primary nodes, for which the spectrum is licensed and secondary (cognitive) nodes, which are the secondary users of spectrum. Here, we consider a special case when primary and secondary nodes are weakly correlated.

A. *Interference Avoidance*

The secondary neighbors of a primary node need to hold back from transmission while the primary node is receiving. This can be implemented, for example, if the primary receiving node transmits a beacon signal. The secondary neighbors which can detect this signal abstain from transmission. The ith nearest secondary neighbor abstains from transmission, with probability ρ_i, if the SNR of the received beacon is larger than a threshold (γ_t). We denote the power of the beacon signal and the noise as P_b and N respectively. The power of the beacon signal decays according to a distance-dependent path loss model. We also consider a channel fading component and denote it as X. ρ_i can, therefore, be calculated as

$$\rho_i = \Pr\left\{\frac{P_b X^2 R^{-\alpha}_{AB,i}}{N} > \gamma_t\right\} = E\left\{F_{c,Y}\left(kR^{\alpha}_{AB,i}\right)\right\}, \quad (17)$$

where $Y = X^2$, $F_{c,Y}(.)$ is the CCDF of Y, $k = \frac{\gamma_t N}{P_b}$ and α is the path loss exponent. For a Rayleigh fading channel with $E\{Y\} = 1$, we have $F_{c,Y}(y) = e^{-y}$. Using (7) and with the weak correlation assumption (i.e., $\chi(r) = (a + bc)r^2$),

$$\begin{aligned}\rho_i &= E\{e^{-kR^{\alpha}_{AB,i}}\}\\ &= \int_0^{\infty} e^{-(kr^{\alpha}+(a+bc)r^2)}\frac{\left[(a+bc)r^2\right]^{i-1}}{(i-1)!}2(a+bc)rdr.\end{aligned}$$

This integral can be found numerically. Convexity of exponential function can also be used to lower bound ρ_i as $\rho_i \geq e^{-kE\{R^{\alpha}_{AB,i}\}}$. Figure 2 is obtained by averaging the simulations after 1000 iterations for $k = 0.005$. The results show that the simulation curve follows the analytical curve closely and also the obtained lower bound is very tight.

B. *Interference Modeling*

One critical issue of concern in cognitive radio networks is the level of interference from the secondary network on primary nodes. The aggregate interference from secondary neighbors is

$$I = \sum_i (1-\rho_i)P_i x_i^2 R^{-\alpha}_{AB,i}. \quad (18)$$

Using the results found in lemma 2, the interference on primary nodes can be characterized by obtaining its moments. Measures like power control can be taken by the secondary network (i.e., by controlling $\{P_i\}$) to avoid the interference on primary nodes beyond a certain threshold [5].

IV. CONCLUSION

We have considered a spatial bivariate Poisson point process to model heterogeneous random networks consisting of two types of nodes. The number of nodes of each type in a region may be correlated. We obtain the distribution of inter-nodal distances and also closed-form results for their real moments in two special cases in which the number of points of the two types in a region are strongly or weakly correlated and verify the accuracy of results using simulation. We provide an application example for cognitive radio networks and assume weak correlation between primary and secondary nodes. We use the results to obtain a tight lower bound for the probability of abstention of secondary nodes at the presence of a primary receiving node. Other applications for interference modeling have also been discussed.

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