Effect of Coordinate Switching on Translunar Trajectory Simulation Accuracy

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This paper focuses on the affect of round-off error in the accurate simulation of translunar trajectories. The three-body dynamics can be posed in either an Earth-centered (EC) or Moon-centered frame. In this study, multiple translunar trajectories were simulated to determine if there is an optimal switch point from an EC to MC frame that minimize round-off error. A high fidelity baseline simulation was first created, and the entire trajectory was propagated in EC. Comparison trajectories were then simulated at lower precision, and the trajectory was propagated first in EC, then switched to MC at varying points in the trajectory. From initial results, the optimal switching can reduce round-off error by more than 50%, and the optimal switch point appears to correspond closely to the Moon's sphere of influence.

Nomenclature

G	Gravitational Constant
T_m	Period of the moon
a_m	Semi-major axis of the moons orbit
m_e	Mass of the earth
m_m	Mass of the moon
$\mathbf{r_e}$	Position vector from the moon to the earth
$\mathbf{r_m}$	Position vector from the earth to the moon
$\mathbf{r_{v_e}}$	Position vector from the earth to the vehicle
$\mathbf{r_{v_m}}$	Position vector from the moon to the vehicle
r_{soi}	Radius of a sphere of influence
r_{hill}	Radius of a Hill sphere
t_0	Initial time
ω_m	Angular velocity of the moon

I. Introduction

NASA's Project Constellation is committed to the long-term human and robotic exploration of the Moon, Mars, and beyond. Although the spacecraft developed will still have the capability of delivering payload to Low Earth Orbit (LEO), there is a renewed emphasis on leaving the gravitational influence of the earth, and traveling to the moon. Compared to the era of the Apollo missions, numerical simulations have grown in capability and importance. As a result, simulation accuracy has also improved significantly and numerical results can now be calculated with much greater precision. Until recently, these more advanced tools have primarily been used in simulating LEO's. The new focus on lunar missions, however, motivates the development of precise translunar simulations.

Pierre-Simon Laplace's seminal work Mécanique Céleste helped transform the study of classical mechanics from a geometry based study to one based on calculus. One topic covered by Laplace was the accurate computation of the motion of a comet which was about to pass near Jupiter.[3] He suggested a patched-conic

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approximation with two different models of the trajectory. The models centered on either the Sun or Jupiter, treating the other body as a perturbation. To switch between these models, Laplace suggested a “sphere of influence” where the ratio of the perturbative to central acceleration in each model is equal. The radius of this approximated sphere is governed by the following equation, written for the bodies of interest in this paper:

$$r_{soi} = a_m \left(\frac{m_m}{m_e}\right)^{2/5} \tag{1}$$

Here, a_m is the orbital radius of the Moon, m_m is the mass of the Moon, and m_e is the mass of the Earth. The value of r_{soi} for the moon is 66,183.46 km. A modified sphere of influence was later proposed by American astronomer George William Hill and is known as a Hill sphere.

$$r_{hill} = a_m \left(\frac{m_m}{3m_e}\right)^{1/3} \tag{2}$$

Its value for the earth moon system is 61,524.49 km

The issue of model choices, affects the numerical error in the simulation of translunar trajectories. Numerical errors typically include several sources such as discretization error, round-off error, and iterative error. This paper will investigate the role of coordinate choice in reducing round-off error to improve simulation accuracy.

II. System Model

The positions of the vehicle, Earth, and Moon are described by the relative position vectors as shown in Fig. 1. In this current work, these vectors are assumed to be coplanar.

$$\mathbf{r_{v_e}} = \mathbf{r_{v_m}} + \mathbf{r_m} \tag{3}$$

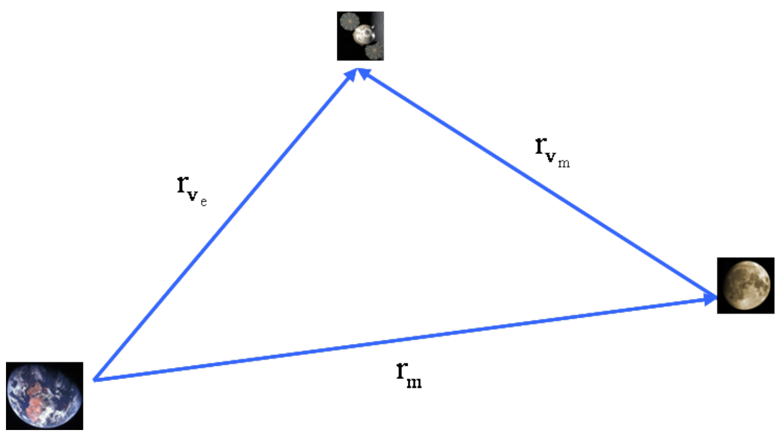

Figure 1. Position vectors between the vehicle, Earth, and Moon.

The relative motion of the vehicle can be described with respect to either the Earth or the Moon. This gives rise to alternative sets of equations of motion in either an Earth-centered (EC) or Moon-centered (MC) reference frame. The right-hand side of both equations contains a central-body acceleration and a perturbative equation. The equations in the EC frame are shown below, where the Earth is the central body and the Moon causes a pertrubative acceleration.

$$\ddot{\mathbf{r}}_{\mathbf{v_e}} = \frac{-Gm_e}{r_{v_e}^3}\mathbf{r_{v_e}} - Gm_m\left(\frac{\mathbf{r_{v_e}} - \mathbf{r_m}}{(r_{v_e} - r_m)^3} + \frac{\mathbf{r_m}}{r_m^3}\right) \tag{4}$$

The equations in the MC frame are similar, however, the Moon is now treated as the central body and the Earth causes the perturbative acceleration.

$$\ddot{\mathbf{r}}_{\mathbf{v_m}} = \frac{-Gm_m}{r_{v_m}^3}\mathbf{r}_{\mathbf{v_m}} - Gm_e\left(\frac{\mathbf{r}_{\mathbf{v_m}} - \mathbf{r_e}}{(r_{v_m} - r_e)^3} + \frac{\mathbf{r_e}}{r_e^3}\right) \tag{5}$$

The position of the Moon was modeled as a circular motion.

$$\mathbf{r_m} = a_m \begin{bmatrix} \cos(\omega_m t_0) \\ \sin(\omega_m t_0) \end{bmatrix} \tag{6}$$

Here, the value of the angular velocity $\omega_m = \frac{2\pi}{T_m}$ was derived from Kepler's third law of planetary motion:

$$\left(\frac{T_m}{2\pi}\right)^2 = \frac{a_m^3}{G(m_e + m_m)} \tag{7}$$

A value of $a_m = 384,400$ km was used as the semi-major axis of the Moon's orbit.

The vehicle trajectory can be simulated by numerical integration of either Eq. (4) or Eq. (5). In this work, fourth-order Runge-Kutta integration will be considered. These contain both discretization and round-off errors; whereas other simulations can also contain iterative error. In the equations of motion, calculation of the perturbative acceleration can include larger amounts of round-off error than the central-body acceleration. This is because, for certain configurations of the bodies, calculation of the perturbative acceleration can require subtraction of similar numbers.

III. Simulation Approach and Numerical Results

In order to investigate the effect of coordinate choice on round-off error, a MATLAB simulation was created that allowed switching between the EC and MC models. This simulation was implemented with single precision computations. To estimate the amount of roundoff error, the results were compared to a "truth" simulation using double precision.

In MATLAB, a floating-point number handled in double precision format uses 64 bits (8 bytes) of memory storage based on IEEE Standard 754. Double-precision variables accurately represent values to approximately 15 decimal places. The lower 32-bit single-precision variables represent data to about seven decimal places.

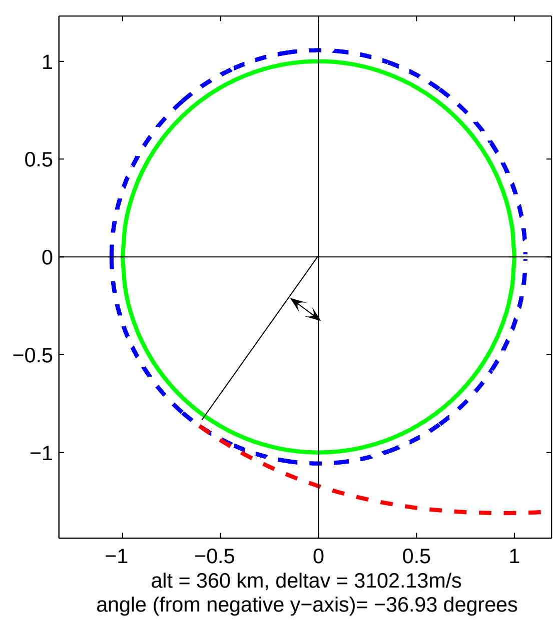

Figure 2. Plot of the earth, low earth orbit, point of Δv, and initial trajectory.

A nominal trip time of 3.5 days was chosen for the trajectory, and an integration time step of 20 secs was used for all simulations. The initial part of the trajectory is illustrated in Fig. 2, and the starting location of the moon is directly out from the earth along the positive x-axis.

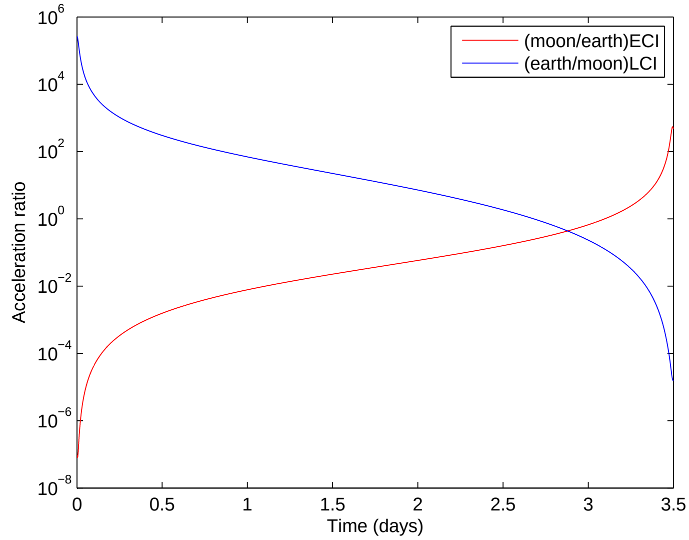

Figure 3. Ratio of perturbative to central-body acceleration.

Figure 3 clearly illustrates how the gravitational influence of the earth dominates the trajectory at first, but as the spacecraft approaches the moon, the gravitational pull of the moon gradually takes over. From a magnified view of this plot in Fig 4, the two curves intersect at 2.88 days which corresponds to a distance of 59,338.35 km from the moon. This value is in close agreement with the value obtained for r_{soi} and r_{hill}.

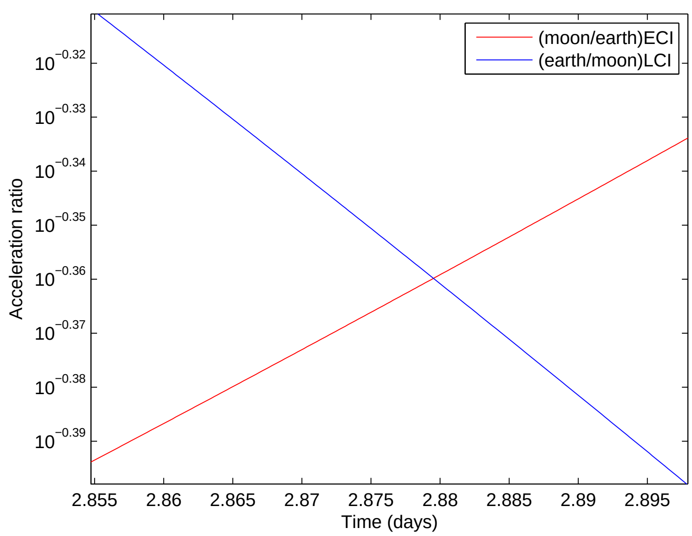

Figure 4. Magnified Ratio of perturbative to central-body acceleration.

This affect can also be seen in Fig. 5 which shows the trajectory of the spacecraft as it is captured by the gravity of the moon.

Figures 6 and 7 show preliminary results for the round-off error using several switch points. The root-sum-squared position errors over the entire trajectory are plotted with respect to time. These results indicate that switching coordinates either very early or very late in the trajectory introduces large round-off errors. These errors are mitigated by selecting a more optimal point at which to switch from EC to MC.

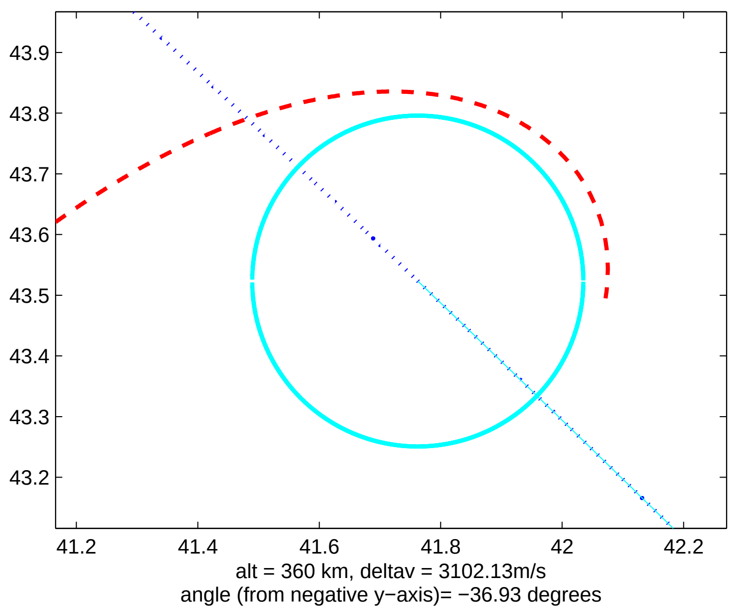

Figure 5. Trajectory of the spacecraft as it is captured by the moon.

From Fig. 6 and 7, it can be seen that the lowest overall error is achieved by switching at around 2.6 days which corresponds to a distance of 59,292.52 km from the moon. This value is also in close agreement with the value obtained for r_{soi} and r_{hill}.

IV. Future Work

The issue studied in this work, comparing high and low precision simulations, is closely related to the development of Chaos theory. For certain systems, small changes in initial conditions can produce significantly different future behaviors. The Lyapunov exponent is one tool that has been developed to investigate system divergence, and future work may investigate Lyapunov exponents along the trajectories. Future work may also further investigate other sources of numerical error in translunar trajectory simulations. Richardson extrapolation can be used to estimate discretization error, and the method of nearby problems can be used to characterize the total numerical error.

V. Conclusion

In the simulation of translunar trajectories, the vehicle motion can be described in either an EC or MC frame. These coordinate choices have significant impact on the amount of round-off error in the simulation. It was shown that the switching from EC to MC coordinates can reduce errors by over 50%. The preferred switch point is closely correlated with the historical concepts of the sphere of influence and Hill's sphere.

Acknowledgments

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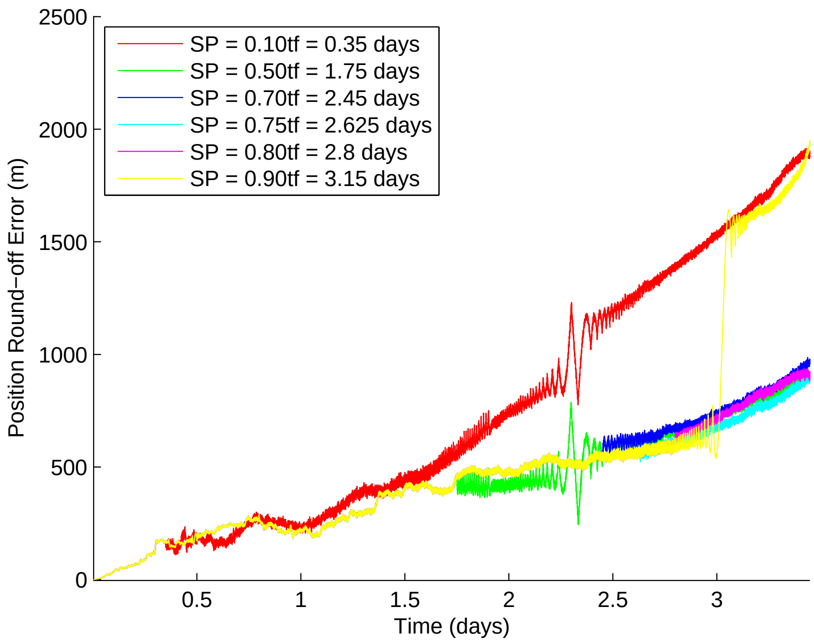

Figure 6. Plot of the RSS position error.

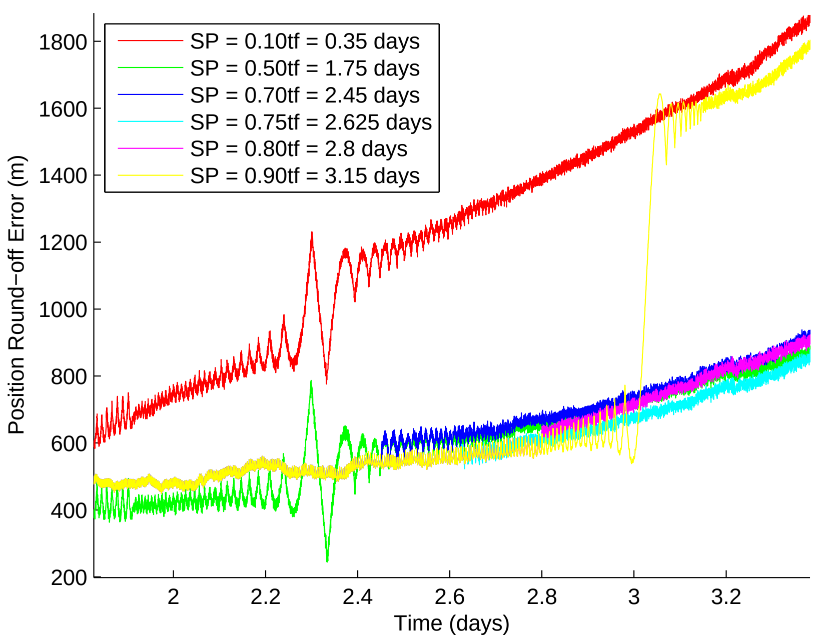

Figure 7. Magnified plot of the RSS position error.