

# 26

## Analog Filter Synthesis

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### 26.1 Introduction

Analog filters are essential in many different systems that electrical engineers are required to design in their engineering career. Filters are widely used in communication technology as well as in other applications. Although we discuss and talk a lot about digital systems nowadays, these systems always contain one or more analog filters internally or as the interface with the analog world [SV01].

There are many different types of filters such as Butterworth filter, Chebyshev filter, inverse Chebyshev filter, Cauer elliptic filter, etc. The characteristic responses of these filters are different. The Butterworth filter is flat in the stop-band but does not have a sharp transition from the pass-band to the stop-band while the Chebyshev filter has a sharp transition from the pass-band to the stop-band but it has the ripples in the pass-band. Oppositely, the inverse Chebyshev filter works almost the same way as the Chebyshev filter but it does have the ripple in the stop-band instead of the pass-band. The Cauer filter has ripples in both pass-band and stop-band; however, it has lower order [W02, KAS89]. The analog filter is a broad topic and this chapter will focus more on the methodology of synthesizing analog filters only (Figures 26.1 and 26.2).

Section 26.2 will present methods to synthesize four different types of these low-pass filters. Then we will go through design example of a low-pass filter that has 3 dB attenuation in the pass-band, 30 dB attenuation in the stop-band, the pass-band frequency at 1 kHz, and the stop-band frequency at 3 kHz to see four different results corresponding to four different synthesizing methods.

### 26.2 Methods to Synthesize Low-Pass Filter

#### 26.2.1 Butterworth Low-Pass Filter

- $\omega_p$ —pass-band frequency
- $\omega_s$ —stop-band frequency
- $\alpha_p$ —attenuation in pass-band
- $\alpha_s$ —attenuation in stop-band

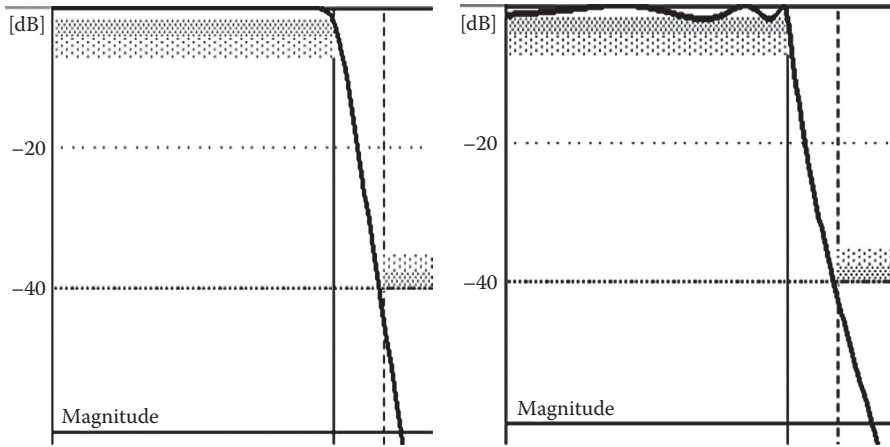


FIGURE 26.1 Butterworth filter (left), Chebyshev filter (right).

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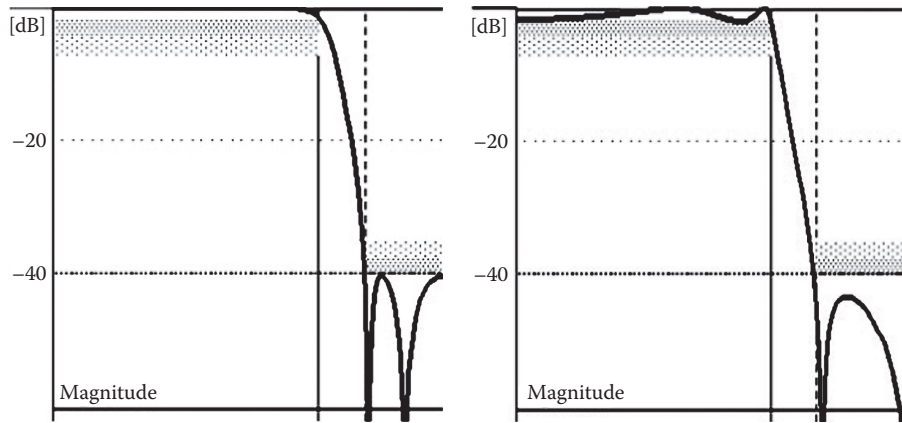


FIGURE 26.2 Inverse Chebyshev filter (left), Cauer elliptic filter (right).

Butterworth response (Figure 26.3):

$$|T(j\omega)|^2 = \frac{1}{1 + (\omega^{2n} / \omega_0^{2n})}$$

There are three basic steps to synthesize any type of low-pass filters. The first step is calculating the order of a low-pass filter. The second step is calculating poles and zeros of a low-pass filter. The third step is design circuits to meet pole and zero locations; however, this part is another topic of analog filters, so it will be not be covered in this work [W90, WG05, WLS92].

All steps to design Butterworth low-pass filter.

*Step 1:* Calculate order of filter:

$$n = \frac{\log[(10^{\alpha_s/10} - 1)(10^{\alpha_p/10} - 1)]^{1/2}}{\log(\omega_s / \omega_p)} \quad (n \text{ needs to be roundup to integer value})$$

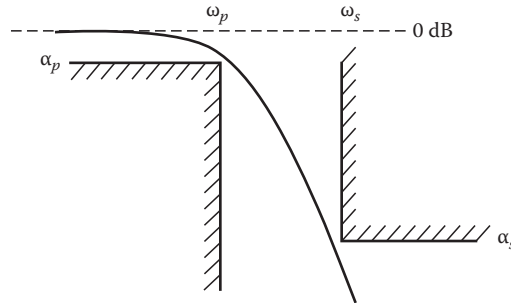


FIGURE 26.3 Butterworth filter characteristic.

Step 2: Calculate pole and zero locations:

Angle if  $n$  is odd:

$$\Omega = \pm \frac{k180^\circ}{n}; \quad k = 0, 1, \dots, \frac{n-1}{2}$$

Angle if  $n$  is even:

$$\Omega = \pm \left( 0.5 + \frac{k}{n} \right) 180^\circ; \quad k = 0, 1, \dots, \frac{n-2}{2}$$

Normalized pole locations:

$$a_k = -\cos(\Omega); \quad b_k = \pm \sin(\Omega); \quad (\omega_0 = 1)$$

$$\omega_0 = \frac{(\omega_p \omega_s)^{1/2}}{[(10^{\alpha_s/10} - 1) / (10^{\alpha_p/10} - 1)]^{1/(4n)}}; \quad Q_k = \left| \frac{1}{2a_k} \right|$$

Step 3: Design circuits to meet pole and zero locations (not covered in this work) (Figure 26.4).

### Example:

Step 1: Calculate order of filter:

$$n = \frac{\log[(10^{30/10} - 1)(10^{3/10} - 1)]^{1/2}}{\log(3000/1000)} = 3.1456 \Rightarrow n = 4$$

Step 2: Calculate pole and zero locations

Normalized values of poles and  $\omega_0$  and  $Q$ :

$-0.38291 + 0.92443i$	1.00059	1.30656
$-0.38291 - 0.92443i$	1.00059	1.30656
$-0.92443 + 0.38291i$	1.00059	0.54120
$-0.92443 - 0.38291i$	1.00059	0.54120

Normalized values of zeros  $\Rightarrow$  none.



Step 2: Calculate pole and zero locations:

$$\Omega = 90^\circ + \frac{90^\circ}{n} + \frac{(k-1)180^\circ}{n}$$
$$\varepsilon = \left[ 10^{\alpha_p/10} - 1 \right]^{1/2}; \quad \gamma = \frac{\sinh^{-1}(1/\varepsilon)}{n}$$
$$a_k = \sinh(\gamma) \cos(\Omega); \quad b_k = \cosh(\gamma) \sin(\Omega); \quad \omega_k = a_k^2 + b_k^2; \quad Q_K = \left| \frac{\omega_k}{2a_k} \right|$$

Step 3: Design circuits to meet pole and zero locations (not covered in this work) (Figure 26.6).

**Example:**

Step 1: Calculate order of filter:

$$n = \frac{\ln[4 * (10^{30/10} - 1) / (10^{3/10} - 1)]^{1/2}}{\log[(3000/1000) + ((3000^2/1000^2) - 1)^{1/2}]} = 2.3535 \Rightarrow n = 3$$

Step 2: Calculate pole and zero locations  
Normalized values of poles and  $\omega_0$  and Q:

$-0.14931 + 0.90381i$	0.91606	3.06766
$-0.14931 - 0.90381i$	0.91606	3.06766
$-0.29862$		

Normalized values of zeros  $\Rightarrow$  none.

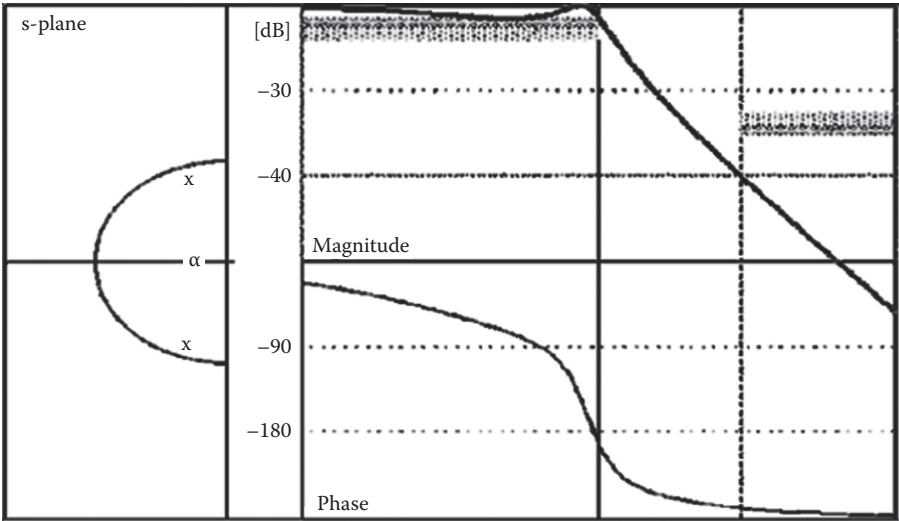


FIGURE 26.6 Pole-zero locations, magnitude response, and phase of Chebyshev filter.

### 26.2.3 Inverse Chebyshev Low-Pass Filter

$\omega_p$ —pass-band frequency  
 $\omega_s$ —stop-band frequency  
 $\alpha_p$ —attenuation in pass-band  
 $\alpha_s$ —attenuation in stop-band

Inverse Chebyshev response (Figure 26.7):

$$|T_{IC}(j\omega)|^2 = \frac{\epsilon^2 C_n^2(1/\omega)}{1 + \epsilon^2 C_n^2(1/\omega)}$$

The method to design the inverse Chebyshev low-pass filter is almost the same as the Chebyshev low-pass filter. It is just slightly different.

*Step 1:* Calculate order of filter

$n$  = order of the Chebyshev filter

*Step 2:* Calculate pole and zero locations:

$$P_{ic} = \frac{1}{a_k + b_k}, \quad \text{find zeros} \quad \omega_i = \frac{1}{\cos[\Pi * i / (2n)]}; \quad i = 2k - 1 : 1, 3, 5 \dots < np$$

*Notes:* two conjugate poles on the imaginary axis.

*Step 3:* Design circuits to meet pole and zero locations (not covered in this work) (Figure 26.8).

#### Example:

*Step 1:* Calculate order of filter:

$$n = \frac{\ln[4 * (10^{30/10} - 1) / (10^{3/10} - 1)]^{1/2}}{\log[(3000/1000) + ((3000^2/1000^2) - 1)^{1/2}]} = 2.3535 \Rightarrow n = 3$$

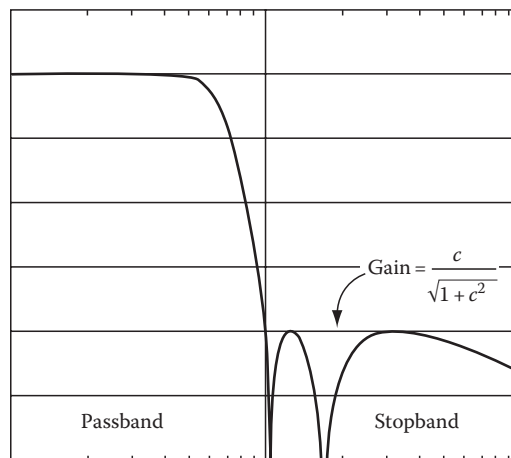


FIGURE 26.7 Inverse Chebyshev filter characteristic.

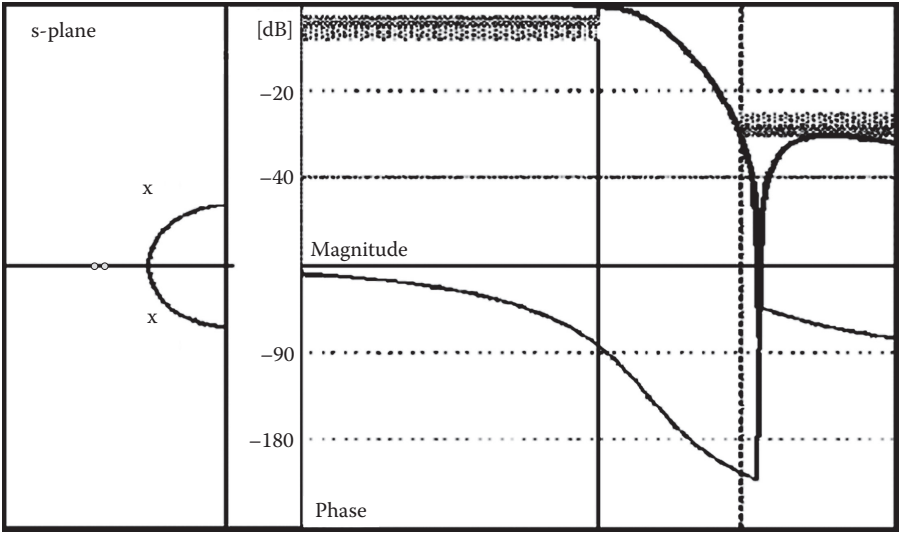


FIGURE 26.8 Pole-zero locations, magnitude response, and phase of inverse Chebyshev filter.

Step 2: Calculate pole and zero locations  
Normalized values of poles and  $\omega_0$  and  $Q$ :

$-0.6613 + 1.29944i$	1.45803	1.10240
$-0.6613 - 1.29944i$	1.45803	1.10240
$-1.60734$		

Normalized values of zeros:

$3.4641i$	$3.4641i$	3.4641
$-3.4641i$	$-3.4641i$	3.4641

### 26.2.4 Cauer Elliptic Low-Pass Filter

Cauer elliptic response (Figure 26.9):

$$|T(jw)|^2 = \frac{1}{1 + \epsilon^2 R_n^2(w, L)}$$

To design the Cauer elliptic filter is more complicated than designing three previous filters. In order to calculate the transfer function of this filter, a mathematic process is summarized as below. Although the low-pass Cauer elliptic filter has ripples in both stop-band and pass-band, it has lower order than the three previous filters (Figure 26.10). That is the advantage of the Cauer elliptic filter:

$$k = \frac{\omega_p}{\omega_s} \tag{26.1}$$

$$k' = \sqrt{1 - k^2} \tag{26.2}$$

$$q_0 = \frac{0.5(1 - \sqrt{k'})}{(1 + \sqrt{k'})} \tag{26.3}$$

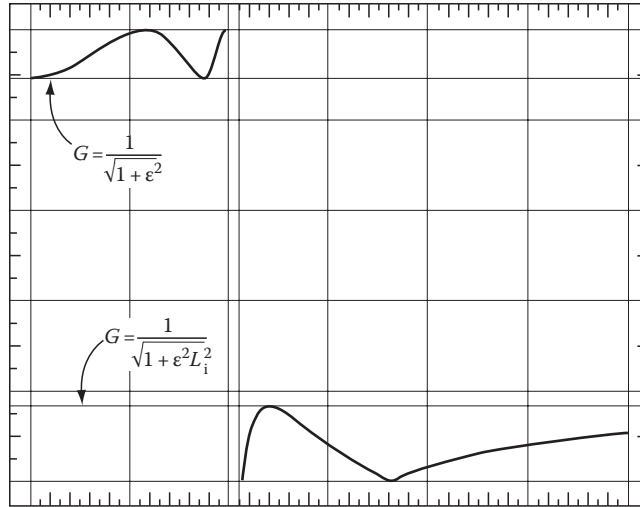


FIGURE 26.9 Cauer elliptic filter characteristic.

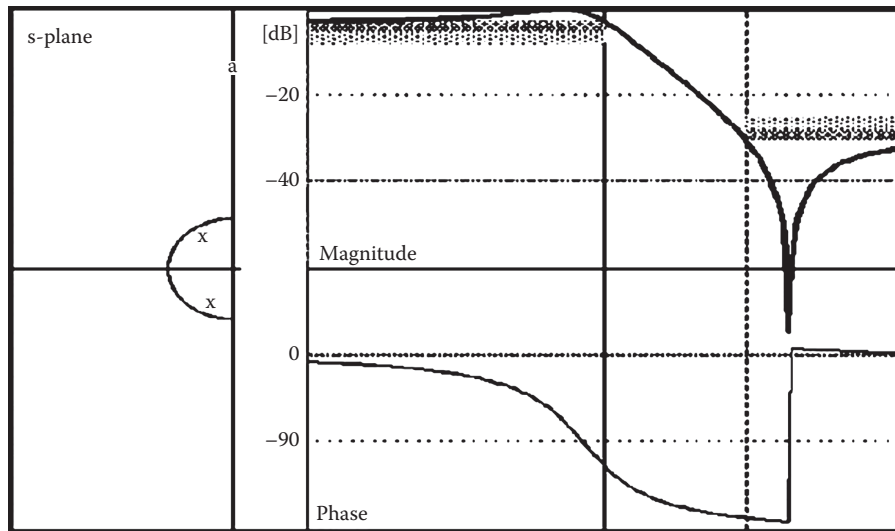


FIGURE 26.10 Pole-zero locations, magnitude response, and phase of Cauer elliptic filter.

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} \quad (26.4)$$

$$D = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \quad (26.5)$$

$$n \geq \frac{\log(16D)}{\log(1/q)} \quad (26.6)$$

$$\Lambda = \frac{1}{2n} \ln \frac{10^{0.05\alpha_p} + 1}{10^{0.05\alpha_p} - 1} \quad (26.7)$$



$$\sigma_0 = \left| \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sinh[(2m+1)\Lambda]}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh(2m\Lambda)} \right| \quad (26.8)$$

$$\omega = \sqrt{\left(1 + k\sigma_0^2\right) \left(1 + \frac{\sigma_0^2}{k}\right)} \quad (26.9)$$

$$\Omega_i = \left| \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sinh((2m+1)\pi\mu/n)}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh\left(\frac{2m\pi\mu}{n}\right)} \right| \quad (26.10)$$

$$\mu = \begin{cases} i & \text{for odd } n \\ i - \frac{1}{2} & \text{for even } n \end{cases} \quad i = 1, 2, \dots, r \quad (26.11)$$

$$V_i = \sqrt{\left(1 - k\Omega_i^2\right) \left(1 - \frac{\Omega_i^2}{k}\right)} \quad (26.12)$$

$$A_{0i} = \frac{1}{\Omega_i^2} \quad (26.13)$$

$$B_{0i} = \frac{(\sigma_0 V_i)^2 + (\Omega_i \omega)^2}{(1 + \sigma_0^2 \Omega_i^2)^2} \quad (26.14)$$

$$B_{1i} = \frac{2\sigma_0 V_i}{1 + \sigma_0^2 \Omega_i^2} \quad (26.15)$$

$$H_0 = \begin{cases} \sigma_0 \prod_{i=1}^r \frac{B_{0i}}{A_{0i}} & \text{for odd } n \\ 10^{-0.05\alpha_p} \prod_{i=1}^r \frac{B_{0i}}{A_{0i}} & \text{for even } n \end{cases} \quad (26.16)$$

**Example:**

$n = 1.9713 \Rightarrow n = 2$ . This filter is the second low pass filter.

Normalized values of poles and  $\omega_0$  and  $Q$ :

$-0.31554 + 0.97313i$	0.85360	1.35259
$-0.31554 + 0.97313i$	0.85360	1.35259

Normalized values of zeros:

$4.18154i$	$4.18154i$	4.18154
$-4.18154i$	$-4.18154i$	4.18154

## 26.3 Frequency Transformations

Four typical methods of deriving a low-pass transfer function that satisfies a set of given specifications are presented. However, there are a lot of applications in the real world of designing, which require not only the low-pass filters but also the band-pass filters, high-pass filters, and band-rejection filters. A designer can design any type of filters by designing a low-pass filter first. When a low-pass filter is achieved, the desired filter can be derived by “frequency transformation.” In other words, the understanding of methods to design a low-pass filter is the basic but not the trivial task.

### 26.3.1 Frequency Transformations Low-Pass to High-Pass

$$Z(s) = S = \frac{1}{s}; \quad j\Omega = \frac{1}{j\omega} \Rightarrow \Omega = \frac{1}{\omega}; \quad \begin{array}{l} -1 \leq \Omega \leq 1 \text{ frequency of low-pass passband} \\ -1 \leq \omega \leq 1 \text{ frequency of high-pass passband} \end{array}$$

Frequency transformation transforms the pass-band of the low-pass, centered around  $\Omega = 0$ , into that of the high-pass, centered around  $\omega = \infty$  (Figure 26.11). Similarly, it transforms the low-pass stop-band that is centered around  $\Omega = \infty$  into that of the high pass, centered around  $\omega = 0$ . Consequently, the frequency transformation function  $Z(s)$  has a zero in the center of the pass-band of the high-pass (at  $\omega = \infty$ ) and a pole in the center of the high-pass, stop-band (at  $\omega = 0$ ) [SV01]:

$$T(S) = \frac{\omega_0^2}{S^2 + (\omega_0 S/Q) + \omega_0^2} \Rightarrow T(s) = \frac{\omega_0^2}{(1/s^2) + (\omega_0/Qs) + \omega_0^2} = \frac{s^2}{(1/\omega_0^2) + (s/\omega_0^2 Q) + s^2}$$

$T(S)$ : low-pass transfer function;  $T(s)$ : high-pass transfer function.

### 26.3.2 Frequency Transformations Low-Pass to Band-Pass

$$Z(s) = S = \frac{s^2 + \omega_c}{Bs} = \frac{\omega_c(s^2 + \omega_c^2)}{B\omega_c s} \Rightarrow \Omega = \frac{\omega^2 - \omega_c^2}{B\omega}; \quad \omega_c^2 = \omega_1\omega_2; \quad B = \omega_2 - \omega_1$$

Frequency transformation transforms the pass-band of the low-pass, centered around  $\Omega = 0$ , into that of the band-pass, centered around  $\omega = \omega_c$ . Similarly, it transforms the low-pass stop-band that is centered around  $\Omega = \infty$  into that of the band-pass, centered around  $\omega = 0$  (Figure 26.12). Consequently,

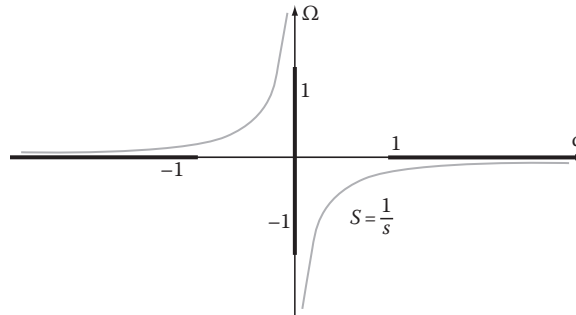


FIGURE 26.11 Frequency transformations low-pass to high-pass.

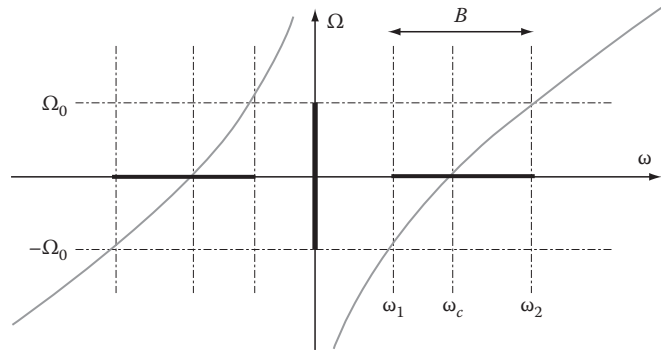


FIGURE 26.12 Frequency transformations low-pass to band-pass.

the frequency transformation function  $Z(s)$  has zeros in the center of the pass-band of the band-pass (at  $\omega = \pm \omega_c$ ) and poles in the center of the band-stop, stop-band (at  $\omega = 0$  and  $\omega = \infty$ ) [SV01]:

$$T(S) = \frac{\omega_0^2}{S^2 + (\omega_0 S/Q) + \omega_0^2} \Rightarrow T(s) = \frac{s^2 B^2 \omega_0^2}{s^4 + (\omega_0 B s^3/Q) + (2\omega_c^2 + B^2 \omega_0^2)s^2 + (\omega_0 B \omega_c^2 s/Q) + \omega_c^4}$$

$T(S)$ : low-pass transfer function;  $T(s)$ : band-pass transfer function.

### 26.3.3 Frequency Transformations Low-Pass to Band-Stop

$$Z(s) = S = \frac{Bs}{s^2 + \omega_c^2} \Rightarrow \Omega = \frac{-B\omega}{\omega^2 - \omega_c^2}; \quad \omega_c^2 = \omega_1 \omega_2; \quad B = \omega_2 - \omega_1$$

Frequency transformation transforms the pass-band of the low-pass, centered around  $\Omega = 0$ , into that of the band-stop, centered around  $\omega = 0$  and  $\omega = \infty$  (Figure 26.13). Similarly, it transforms the low-pass, stop-band that is centered around  $\Omega = \infty$  into that of the band-stop, centered around  $\omega = \omega_c$ . Consequently, the frequency transformation function  $Z(s)$  has zeros in the center of the pass-band of the band-stop (at  $\omega = 0$  and  $\omega = \infty$ ) and poles in the center of the band-stop, stop-band (at  $\omega = \pm \omega_c$ ) [SV01]:

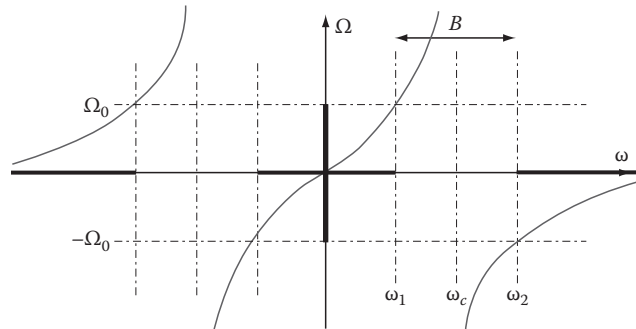


FIGURE 26.13 Frequency transformations low-pass to band-stop.

$$T(S) = \frac{\omega_0^2}{S^2 + (\omega_0 S/Q) + \omega_0^2} \Rightarrow T(s) = \frac{\omega_0^2 s^4 + 2\omega_0^2 \omega_c^2 s^2 + \omega_0^2 \omega_c^4}{\omega_0^2 s^4 + (\omega_0 B s^3/Q) + (2\omega_c^2 \omega_0^2 + B^2) s^2 + (\omega_0 B \omega_c^2 s/Q) + \omega_c^4 \omega_0^2}$$

### 26.3.4 Frequency Transformation Low-Pass to Multiple Band-Pass

Frequency transformation transforms the pass-band of the low-pass, centered around  $\Omega = 0$ , into that of the multiple band-pass, centered around  $\omega = 0$  and  $\omega = \omega_{z1}$ . Similarly, it transforms the low-pass, stop-band that is centered around  $\Omega = \infty$  into that of the multiple band-pass, centered around  $\omega = \omega_{p1}$  and  $\omega = \infty$ . Consequently, the frequency transformation function  $Z(s)$  has zeros in the center of the pass-band of multiple band-pass and at  $\omega_{z1}$  (at  $\omega = 0$  and  $\omega = \pm\omega_{z1}$ ) and poles in the center of the band-stop of multiple pass-band (at  $\omega = \pm\omega_c$  and  $\omega = \infty$ ) [SV01] (Figure 26.14):

$$Z(s) = S = \frac{s(s^2 + \omega_{z1}^2)}{B(s^2 + \omega_{p1}^2)} \Rightarrow \Omega = \frac{\omega(\omega^2 - \omega_{z1}^2)}{B(\omega^2 - \omega_{p1}^2)}$$

Transfer functions from the low-pass frequency  $S$  to the frequency  $s$  of other types of filters are recognized and can be written under the following form:

$$Z(s) = \frac{H(s^2 + \omega_{z1}^2)(s^2 + \omega_{z2}^2) \dots (s^2 + \omega_{zn}^2)}{(s^2 + \omega_{p1}^2)(s^2 + \omega_{p2}^2) \dots (s^2 + \omega_{pn}^2)}$$

$$\text{Or } \Omega(\omega) = \frac{H(\omega^2 - \omega_{z1}^2)(\omega^2 - \omega_{z2}^2) \dots (\omega^2 - \omega_{zn}^2)}{(\omega^2 - \omega_{p1}^2)(\omega^2 - \omega_{p2}^2) \dots (\omega^2 - \omega_{pn}^2)}$$

$Z(s)$  has zeros where the desired filter has pass-bands and poles where it has stop-bands. The function  $Z(s)$  is called Foster Reactance function. For example, we can write the transfer function of the filter (Figure 26.15) as

$$Z(s) = \frac{Hs(s^2 + \omega_z^2)}{(s^2 + \omega_p^2)} \quad \text{or} \quad \Omega(\omega) = \frac{H\omega(\omega^2 - \omega_z^2)}{(\omega^2 - \omega_p^2)}$$

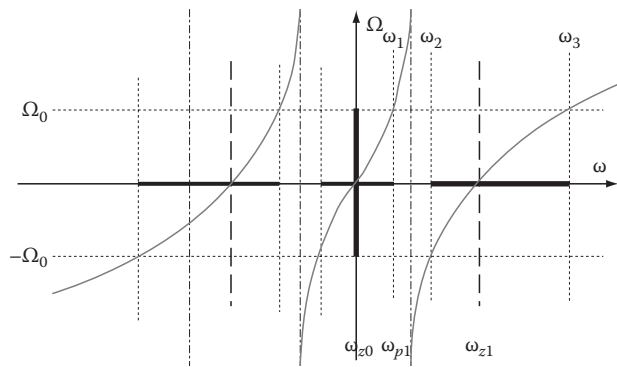


FIGURE 26.14 Frequency transformation low-pass to multiple band-pass.

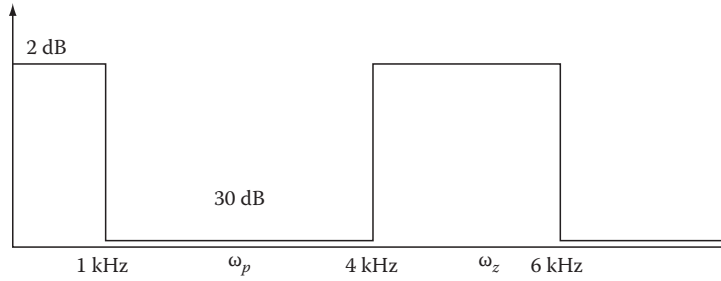


FIGURE 26.15 Frequency transformation by foster reactance function.

The transfer function has zeros at  $\omega = 0$ ,  $\omega = \omega_z$  and poles at  $\omega = \omega_p$  and  $\omega = \infty$ .

At corner frequencies  $\omega_1 = 1$  kHz,  $\omega_2 = 4$  kHz,  $\omega_3 = 6$  kHz, the values of  $\Omega(\omega)$  are equal to 1, -1, and 1, respectively. Therefore, the transformation  $\Omega(\omega)$  can be rewritten into multi-equations corresponding to  $\omega = \omega_1, \omega_2, \omega_3$ . Three equations with three unknowns always have solutions:

$$1 = \frac{H\omega_1(\omega_1^2 - \omega_z^2)}{\omega_1^2 - \omega_p^2}$$

$$-1 = \frac{H\omega_2(\omega_2^2 - \omega_z^2)}{\omega_2^2 - \omega_p^2}$$

$$1 = \frac{H\omega_3(\omega_3^2 - \omega_z^2)}{\omega_3^2 - \omega_p^2}$$

$$\begin{cases} \omega_z^2 = 22 \\ \omega_p^2 = 8 \text{ ; so the Foster Transfer Function is } S = \frac{(1/3)s^3 + (22/3)s}{s^2 + 8} \\ H = \frac{1}{3} \end{cases}$$

## 26.4 Summary and Conclusion

Analog filters have been used broadly in communication. Understanding the methods to synthesizing analog filters is extremely important and is the basic step to design analog filters. Four different synthesizing methods were presented, each method will result in different characteristics of filters. Besides that, this chapter also presented steps to design other types of filters from the low-pass filter by writing the frequency transfer function.

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