GRID REDUCTION ALGORITHM FOR SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In the work reported here, a computationally efficient algorithm for the solution of elliptic partial differential equations on rectangular domains is given. The algorithm uses the "strong implicit" approach and the solution is based on the successive reductions of the grid pattern size. The algorithm requires only four to five iteration steps to reach a solution.

INTRODUCTION

Elliptic partial differential equations are used to solve many engineering problems, such as current flow in distributed systems, thermal conduction, and electrostatic field analysis. These equations are essential for analysis of semiconductor devices and for analysis of parasitic effects in high speed electronic circuits. An algorithm will be illustrated for the solution of the well-known Poisson equation

$$g \nabla^2 V = Q(x,y) \qquad (1)$$

where $V(x,y)$ is the scalar field variable, $Q(x,y)$ represents a source term, and the parameter g characterizes a medium property. In the thermal conduction problem, $V(x,y)$ corresponds to temperature, $Q(x,y)$ represents the power source, and g is the thermal conductivity. In the electrostatic problem, $V(x,y)$, $Q(x,y)$, and g correspond to voltage, charge and the dielectric constant, respectively.

The numerical solution of such problems, using the finite difference representation of the partial derivatives, results in a problem where the repeated solution of a set of linear equations in matrix form:

$$A X = Y \qquad (2)$$

A is an $n \times n$ square matrix, V is a unknown n-vector, and Q is a known n-vector on the right side of the equation.

For any practical problem, the number of linear equations to be simultaneously solved exceeds 1000 for two dimensions and 10000 for three dimensions. Such large systems of equations can not be solved by traditional methods; many special approaches are used instead (Selberherr 1984) (Ortega and Rheinboldt 1970). The simplest is the "relaxation method" which usually requires a few hundred or more iterations. To reduce the computational effort by two or three times, the algorithm is often modified by introducing the "overrelaxation" parameter. A more efficient and more complicated algorithm, uses the "strongly implicit" approach for the solution of matrix equation (2). The following formula is used in an iterative procedure for finding unknown vector V (Selberherr 1984)(Stone 1968)(Wilamowski and Jacquot 1991):

$$V_n = V_p + A_a^{-1} (Q - A V_p) \qquad (3)$$

V_p and V_n, respectively, are the previous and the new values of the unknown vector V during the iteration procedure. The matrix A_a is an approximation to the matrix A, and is chosen in such a way that it can be easily inverted. In equation (3), the inversion of matrix A_a is equivalent to finding an approximate solution. Many different algorithms for approximate solutions are known (Selberherr 1984). Some methods use the matrix inversion technique where certain elements are neglected in order to preserve memory space and shorten the computation time. The best known algorithm, developed by Stone (Stone 1968), usually requires about 20 iterations to achieve an accuracy similar to that of the relaxation method's thousand iterations.

Another method, known as the "cross-type" algorithm (Wilamowski and Jacquot 1991), requires about 8-10 iterations for the same accuracy. The algorithm presented herein uses a new method of finding the approximate solution, and it is more effective than others. Using this new algorithm, a solution can be found in four or five iteration steps.

FUNDAMENTALS OF THE ALGORITHM

In order to achieve rapid convergence using the iteration procedure given by (3), the approximate solution should be as accurate as possible. This accuracy is achieved by first finding the solution for a less dense grid pattern, and then repeatedly transforming the grid into smaller sections. It should be emphasized that for each grid, the implicit solution can be easily found by using the equation derived from Figure 1:

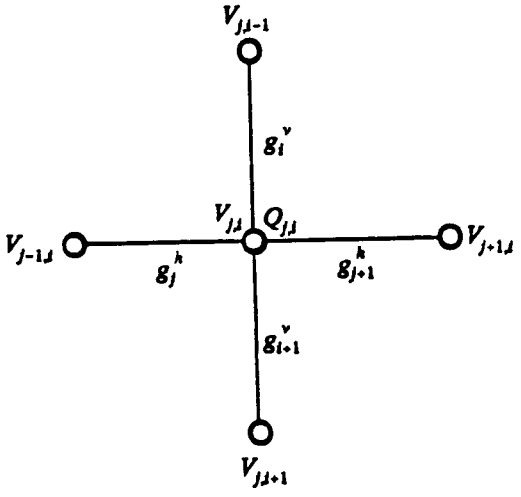

Fig. 1 Single node of computation grid

$$V_{\mu\mu} = \frac{Q_{\mu\mu} + g_i^v V_{\mu-1} + g_{j-1}^h V_{j-1,j} + g_{i+1}^v V_{\mu+1} + g_j^h V_{j-1,\mu}}{g_i^v + g_{j-1}^h + g_{i+1}^v + g_j^h} \quad (4)$$

The error in the final solution is caused by the large grid size for the first few steps. The principle of the grid reduction technique is illustrated in Figure 2. The essence of this approach is the use of rectangular and rhomboid grids commutatively.

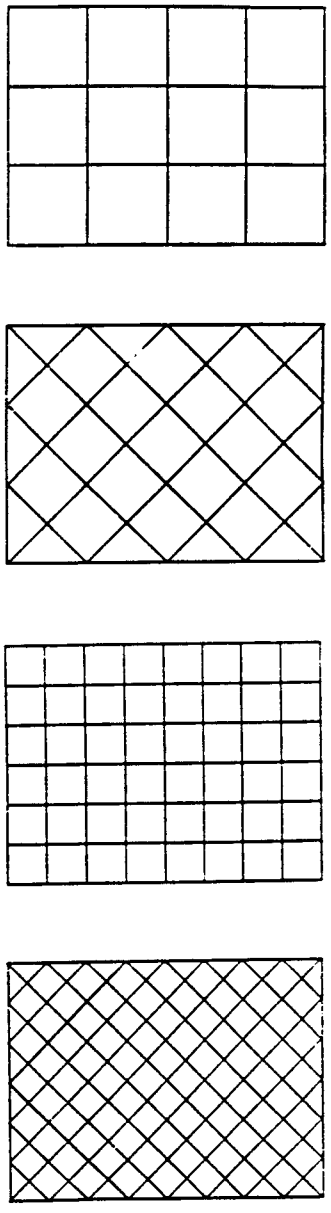

Fig. 2 Illustration of the grid density increase

The algorithm for the solution requires the following steps:

1. Initially, the *g* parameters for all possible grid sizes have to be found. This procedure is started from the most dense grid pattern. The values of the *g* parameters for larger nets are found using the algorithm illustrated in Figure 3.

This computation is done once prior to the iteration procedure using equation (3). For the nonlinear systems where the g parameters are a function of node potential, new values of g parameters have to be found in each iteration step.

2. The charges $Q(x,y)$ associated with each node for each grid size have to be computed. This process starts from the most dense grid pattern. The density of the grid pattern is reduced by splitting the charge of the eliminated nodes among the neighboring nodes proportional to their g parameters as shown in Figure 3. This process of charge computation is required for each iteration step in equation (3).

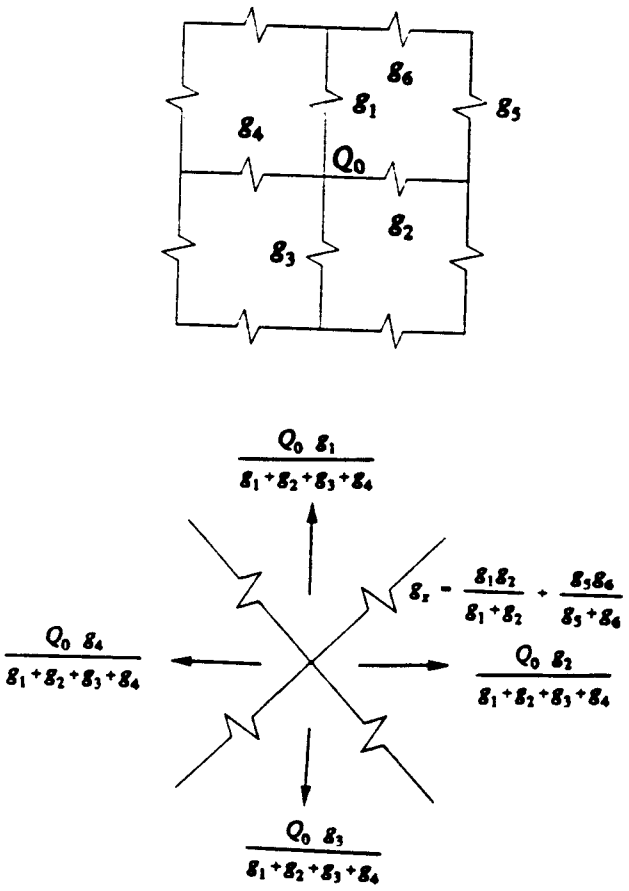

Fig. 3. Parameter transformation between grids of different sizes.

3. Computation of the voltage distribution starts from the less dense grid and proceeds for denser grids using equation (4) as illustrated in Figure 1. It is recommended to start with a reasonably dense grid, for which the implicit algorithm will not require significant computational effort. For example, for the 3 x 4 grid shown in Figure 2, only the voltages of six nodes have to be found in the first step. Note, that many shapes of the rectangular areas can be solved using the proposed approach.
4. Based on the approximate solution of voltages V, the approximated charges at each node can be computed using the equation:

$$Q_{j,i} = \left(g_i^v + g_{j+1}^h + g_{i+1}^v + g_j^h\right) V_{j,i} + \\ - \left(g_i^v V_{j,i-1} + g_{j+1}^h V_{j+1,i} + g_{i+1}^v V_{j,i+1} + g_j^h V_{j-1,i}\right) \quad (5)$$

This procedure corresponds to the matrix multiplication AV in equation (2).

5. Computed values of Q_a are now subtracted from the initial values of Q corresponding to the computation of $(Q - A\,V)$ term in the equation (2)
6. Incremental values of ΔQ obtained in step 5 are now used in step 2. Steps: 2, 3 4, and 5 are repeated iteratively. Since the incremental values are used in the iteration procedure, step 4 is modified so that the voltages from previous steps are corrected by adding the incremental values of ΔV obtained in step 3.

COMPUTATIONAL RESULTS

The result of the algorithm is illustrated with the simple case of a charge source located near one corner of square shape. The results of the computations for each iteration step are shown in Figure 4. Notice the very rapid convergence. The solution is reached after approximately four iteration steps. For the same example and the same accuracy, the "cross-type" algorithm (Wilamowski and Jacquot 1991) required seven iteration steps and the Stone algorithm (Stone 1968) reached a solution after 14 iteration steps. The relaxation method requires about 300 iteration steps.

Figure 5 presents the microscale errors $e_{j,i}$ defined by equation (6) which would eventually be zero at each node for a pure finite difference approach. Note that the errors in Figure 5 are drawn using a different scale for each iteration step. In the example presented for the purpose of demonstration, the distribution used was very simple. When more complicated source distributions were used, the convergence was equally rapid.

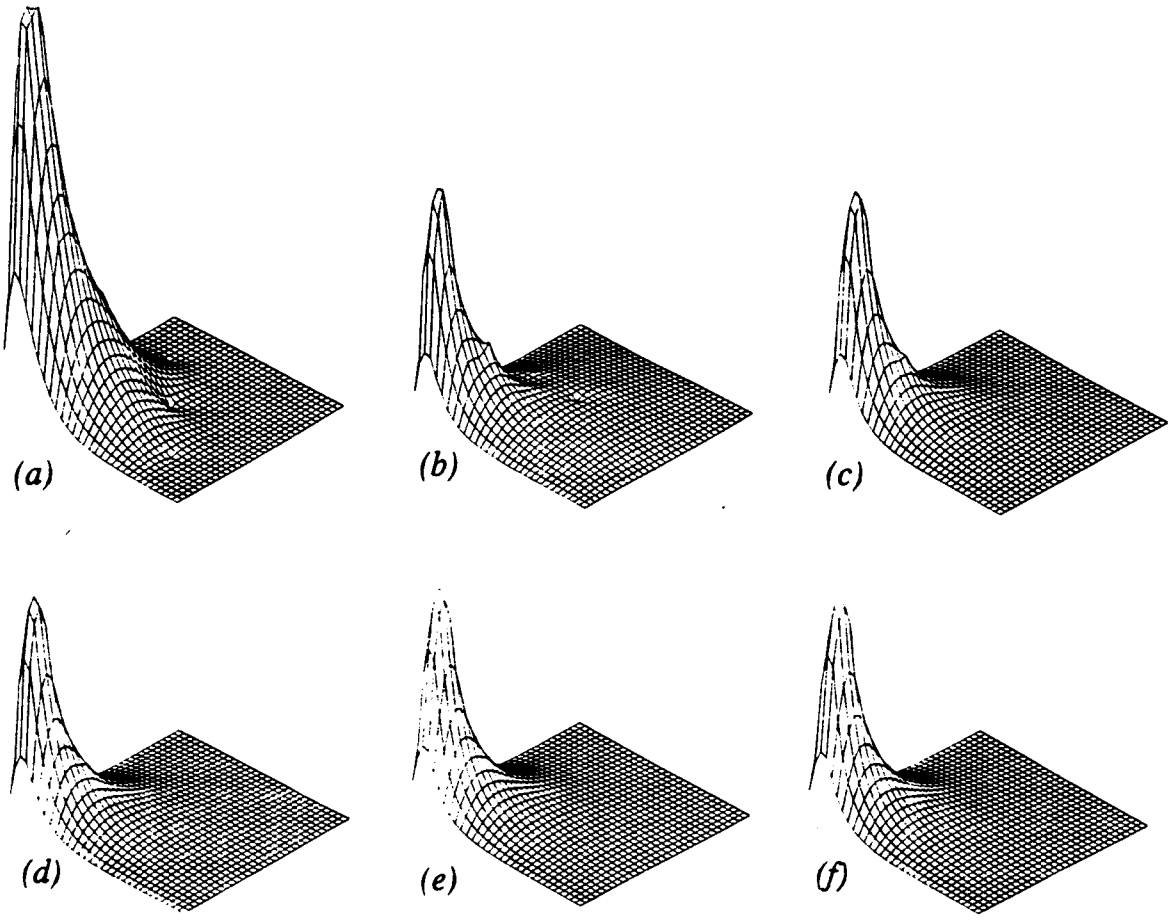

Fig. 4 Scalar field distribution obtained for the first six iteration steps

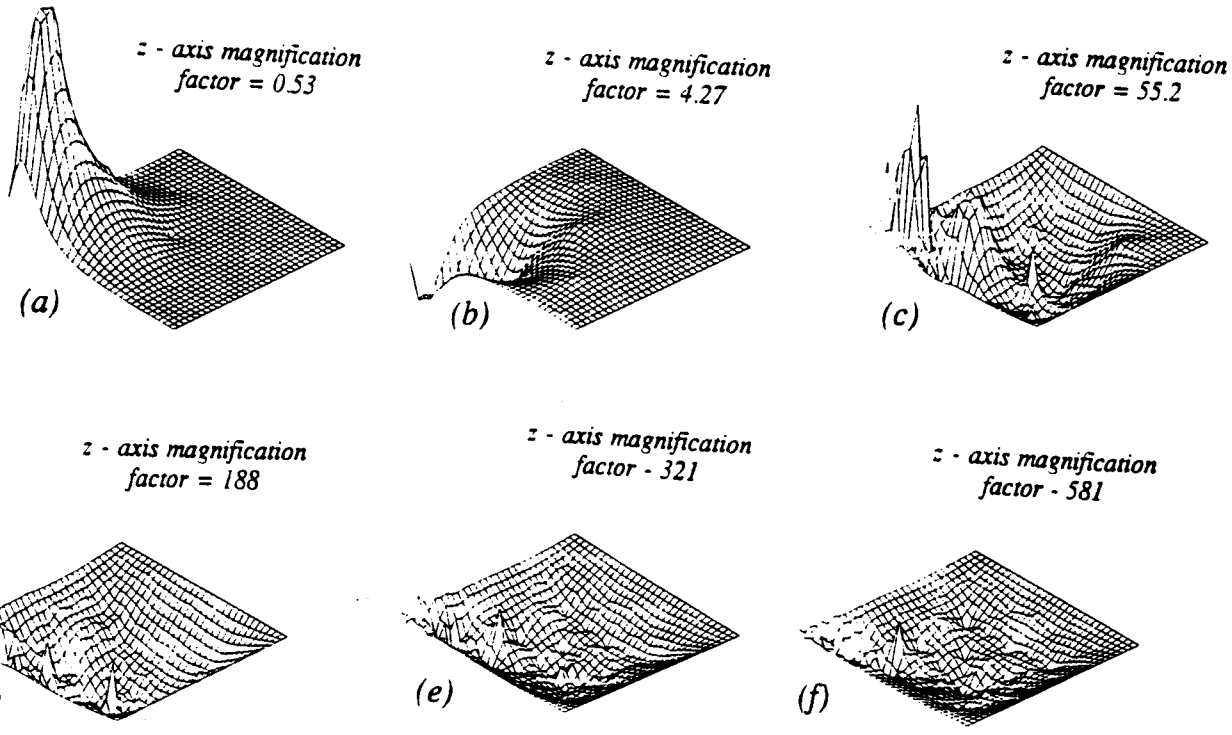

Fig. 5 Microscale errors defined by equation (6) drawn with different scales for the first six iteration steps.

$$e_{ij} = \frac{V_{i-1,j}\, g_{ij}^{h} + V_{i-1,j}\, g_{i-1,j}^{h} + V_{i,j-1}\, g_{ij}^{v} + V_{i,j-1}\, g_{i,j-1}^{v} - Q_{ij}}{g_{ij}^{h} + g_{ij}^{v} + g_{i-1,j}^{h} + g_{i,j-1}^{v}} - V_{ij} \quad (6)$$

The method presented here is very efficient for homogenous materials and seems to be very suitable for solving the Poisson equation for the numerical analysis of semiconductor devices (Greenfield and Dutton 1980)(Wilamowski, Staszak and Mattson 1992). The method was also tested for cases with nonhomogeneous materials, where it was assumed that the medium property g differed by two orders of magnitude. In these cases, the number of required iteration steps increased by a factor of 3 to 5. It should be noted that the presented algorithm can be easily extended for three dimensional structures. Also, in the case of nonhomogeneous mediums, the variable size grid can be implemented without any significant drawback in computational effort.

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